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Quantization of Gauge Field Theories on the Front-Form without Gauge Constraints III: The Schwinger Model[⋆]

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ABSTRACT

In this paper we investigate the implication of a newly proposed method of quantization on the front-form as applied to the Schwinger model. As observed in the previous work, quantizing along both x^+ and x^- , gauge theories does not require extra constraints (also known as 'gauge conditions') to determine the solution uniquely. In this paper we investigate the one-loop corrections to the fermion and photon propagators.

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1. Introduction

In a series of recent papers, we have introduced a new way of quantizing field theories on the front-form, a method in which both x^+ and x^- are treated equally [1] [2] [3] [4] . Penrose and Rindler [5] point out that in this approach there are **no** constraints . In this paper we study this approach applied to the Schwinger model. In particular, we study one-loop corrections to the fermion and photon mass .

There has been a lot of work on the Schwinger model [6] (see also the many references in [7]). I want to focus here on the work done recently on the light-front [8] [9] . The recent treatments of the Schwinger model on the light-front require quantizing with a different procedure than one would use in a general field theory like QED in $3+1$-dimensions: either one used something like the Dirac procedure [9], or one used something akin to our method [8], quantizing on both $x^+ = 0$ and $x^- = 0$ surfaces. There is an economy of means in the (Penrose) quantization used here in that we use the exact same method for the Schwinger model that we used for QED in $3+1$-dimensions. Furthermore, the result we get agrees quite well with the usual canonical result. In a sense, the two-dimensional limit is 'smooth.'

2. Reduced Phase Space Quantization of QED_{1+1}

We follow the reduced phase space quantization of Faddeev and Jackiw [10] (see also [11]) to study QED_{1+1} (we'll keep the fermion mass for now and only take the $m \rightarrow 0$ limit at the end):

$$\mathcal{L} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\partial^\nu\overline{\psi}\gamma_\nu\psi - \overline{\psi}\frac{i}{2}\partial^\nu\gamma_\nu\psi - m\overline{\psi}\psi + \mathcal{L}_I \tag{2.1}$$

where $\mathcal{L}_I$ is the interaction part of the Lagrangean:

$$\mathcal{L}_I = -e\overline{\psi}\gamma^\nu \psi A_\nu \tag{2.2}$$

The Euler-Lagrange equations of motion are

$$\partial_\mu F^{\mu\nu} - e\overline{\psi}\gamma^\nu \psi = 0 \tag{2.3}$$

$$(i\gamma_\mu \partial^\mu - m - e\gamma_\mu A^\mu)\psi = 0 \tag{2.4}$$

To obtain the Hamiltonians for evolution along x^+ and x^-, we follow the approach of previous papers [1], [4], [2] and [3] and we write $\mathcal{L}$ out explicitly :

$$\mathcal{L}d^2x = \{ - F_{+-}F^{+-} - 2e\psi_+^\dagger A_+\psi_+ - 2e\psi_-^\dagger A_-\psi_- + \psi_+^\dagger \frac{i\partial^-}{2}\psi_+ - \frac{i}{2}(\partial^-\psi_+^\dagger)\psi_+$$

$$+\psi_-^\dagger \frac{i\partial^+}{2}\psi_- - \frac{i}{2}(\partial^+\psi_-^\dagger)\psi_- + m\psi_+^\dagger \frac{\gamma^-}{2}\psi_- - m\psi_-^\dagger \frac{\gamma^+}{2}\psi_+\} \frac{dx^-dx^+}{2} \tag{2.5}$$

where $\psi_\pm = \Lambda_\pm \psi$ and $\Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm$. Note also that $\partial_\nu = \frac{\partial}{\partial x^\nu}$ so that $\partial^- = 2\partial_+ = 2\frac{\partial}{\partial x^+}$ and $\partial^+ = 2\partial_- = 2\frac{\partial}{\partial x^-}$. The corresponding conjugate momenta for x^+-derivatives are

$$\pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial^-\psi)} = \frac{i}{2}\psi_+^\dagger \tag{2.6}$$

$$\pi_{\psi^\dagger}(x) = \frac{\partial \mathcal{L}}{\partial(\partial^-\psi^\dagger)} = -\frac{i}{2}\psi_+ \tag{2.7}$$

For the momenta corresponding to x^--derivatives we get similar forms:

$$\rho_\psi(x) = \frac{\partial \mathcal{L}}{\partial(\partial^+\psi)} = \frac{i}{2}\psi_-^\dagger \tag{2.8}$$

$$\rho_{\psi^\dagger}(x) = \frac{\partial \mathcal{L}}{\partial(\partial^+ \psi^\dagger)} = -\frac{i}{2}\psi_- \tag{2.9}$$

We rewrite $\mathcal{L}d^2x$ in the following way

$$\mathcal{L}d^4x = \frac{1}{2}\{\pi_\psi \partial^- \psi_+ + \pi_\psi \partial^- \psi_+ + (\partial^- \psi_+^\dagger)\pi_{\psi^\dagger} + (\partial^- \psi_+^\dagger)\pi_{\psi^\dagger}$$

$$+\rho_\phi \partial^+ \phi + \rho_\phi \partial^+ \phi + \rho_\psi \partial^+ \psi_- + \rho_\psi \partial^+ \psi_- + (\partial^- \psi_-^\dagger)\rho_{\psi^\dagger} + (\partial^- \psi_-^\dagger)\rho_{\psi^\dagger}\}d^2x$$

$$-\mathcal{H}d^2x - \mathcal{K}d^2x + \mathcal{M}d^2x \tag{2.10}$$

The meaning of these terms is as before [2]: the 'p-q-dot' terms which go into the definitions of the canonical commutation relations; the second and third term are the Hamiltonians which define the evolution of the system along x^+ , given by $\mathcal{H}$, and along x^- given by $\mathcal{K}$; finally, the last term contains the remaining pieces which give the 'constraints', though are not 'true constraints' [10] as are consistent with the gauge field equations of motions.

The Hamiltonian densities $\mathcal{H}$ and $\mathcal{K}$ are :

$$\mathcal{H} = \{2e\psi_+^\dagger A_+ \psi_+ + 2e\psi_-^\dagger A_- \psi_- + m\psi_+^\dagger \frac{\gamma^-}{4}\psi_- + m\psi_-^\dagger \frac{\gamma^+}{4}\psi_+\} \tag{2.11}$$

$$\mathcal{K} = \{2e\psi_-^\dagger A_- \psi_- + 2e\psi_+^\dagger A_+ \psi_+ + m\psi_-^\dagger \frac{\gamma^+}{4}\psi_+ + m\psi_+^\dagger \frac{\gamma^-}{4}\psi_-\}. \tag{2.12}$$

and for the 'constraints' we get whatever is left over

$$\mathcal{M} = \{ - F_{+-}F^{+-} + 2eA_+\psi^\dagger_+\psi_+ + 2eA_-\psi^\dagger_-\psi_- \} \tag{2.13}$$

Well, we can rewrite is as (up to total derivatives [10]):

$$\mathcal{M} = A_+C_\pi + A_-C_\rho \tag{2.14}$$

and the 'constraints' C_π and C_ρ are

$$C_\pi = -\partial_- F^{-+} + 2e\psi^\dagger_+\psi_+ \tag{2.15}$$

$$C_\rho = -\partial_+ F^{+-} + 2e\psi^\dagger_-\psi_- \tag{2.16}$$

Let us write the $\mathcal{L}dx^2$ with the explicit momenta dependence (up to total derivatives which we can discard [10], [12]), so as to make the resulting commutation relation clear , and let us set $\mathcal{M} = 0$ as before [2],

$$\mathcal{L}d^2x = \frac{1}{2}2\{\pi_\psi d\psi_+ - d\pi_\psi\psi_+ + d\psi^\dagger_+\pi_{\psi^\dagger} - \psi^\dagger_+ d\pi_{\psi^\dagger}\}\frac{dx^-}{2}$$

$$+\frac{1}{2}2\{\rho_\psi d\psi_- - d\rho_\psi\psi_- + d\psi^\dagger_-\rho_{\psi^\dagger} - \psi^\dagger_- d\rho_{\psi^\dagger}\}\frac{dx^+}{2}$$

$$-\mathcal{H}d^2x - \mathcal{K}d^2x \tag{2.17}$$

We see now that we have **two** types of evolutions: one along x^+, for which the first term in equation (2.17) gives the commutation relations along surfaces $x^+ = y^+$

according to the form:

$$[\xi^a, \xi^b] = \Gamma^{-1}_{ab} \quad a, b = 1, ..4 \tag{2.18}$$

with

$$\xi^1 = \pi_\psi, \xi^2 = \pi_{\psi^\dagger}, \xi^3 = \psi_+, \xi^4 = \psi_+^\dagger \tag{2.19}$$

and

$$\Gamma_{13} = \Gamma_{24} = 2 = -\Gamma_{31} = -\Gamma_{42} \tag{2.20}$$

and all the other two-form Γ's are 0 .

The other evolution is along x^-, for which the second term in equation (2.17) gives the commutation relations along surfaces $x^- = y^-$ according to the form :

$$[\eta^a, \eta^b] = \Delta^{-1}_{ab} \quad a, b = 1, ..4 \tag{2.21}$$

with

$$\eta^1 = \pi_\psi, \eta^2 = \pi_{\psi^\dagger}, \eta^3 = \psi_-, \eta^4 = \psi_-^\dagger \tag{2.22}$$

and

$$\Delta_{13} = \Delta_{24} = 2 = -\Delta_{31} = -\Delta_{42} \tag{2.23}$$

and all the other two-form Δ's are 0 . Going now to the quantum commutators, we get the following relations for fields at equal $x^+ = y^+$, the usual front-form 'time' :

$$\left\{\psi_+(x^+, x^-), \pi_\psi(y^+, y^-)\right\}_{x^+=y^+} = +\frac{i}{2}\Lambda_+\delta(x^- - y^-) \tag{2.24}$$

$$\left\{\psi_+^\dagger(x^+, x^-), \pi_{\psi^\dagger}(y^+, y^-)\right\}_{x^+=y^+} = -\frac{i}{2}\Lambda_+\delta(x^- - y^-) \tag{2.25}$$

Thus, the physical (quantized) degrees of freedom on $x^+ = 0$ are ψ_+ and $\psi_+^\dagger$.

For fields at equal $x^- = y^-$, a new front-form 'time', we get:

$$\left\{\psi_-(x^+, x^-), \rho_\psi(y^+, y^-)\right\}_{x^-=y^-} = +\frac{i}{2}\Lambda_-\delta(x^+ - y^+) \tag{2.26}$$

$$\left\{\psi_-^\dagger(x^+, x^-), \rho_{\psi^\dagger}(y^+, y^-)\right\}_{x^-=y^-} = -\frac{i}{2}\Lambda_-\delta(x^+ - y^+) \tag{2.27}$$

Here, the physical (quantized) degrees of freedom on $x^- = 0$ are ψ_- and $\psi_-^\dagger$. Note that A^+ and A^- do not enter in the list of physical (quantized) degrees of freedom.

Rather, they are obtained as in the previous paper [2], requiring that $\mathcal{M}$ be zero on the initializing surfaces. This gives the following for fields on $x^+ = 0$

$$A_P^+ = 0 \tag{2.28}$$

$$-\frac{1}{2}(\partial^+)^2 A_P^- = 2e\psi_+^\dagger\psi_+ \tag{2.29}$$

On $x^- = 0$ we have

$$A_M^- = 0 \tag{2.30}$$

$$-\frac{1}{2}(\partial^-)^2 A_M^+ = 2e\psi_-^\dagger\psi_- \tag{2.31}$$

Inverting these equations is rather straight-forward, like in the 3+1-dimensional case [2]:

$$A_P^- = \frac{4e}{(i\partial^+)^2}\psi_+^\dagger\psi_+ \tag{2.32}$$

and

$$A_M^+ = \frac{4e}{(i\partial^-)^2}\psi_-^\dagger\psi_- \tag{2.33}$$

To fully define these fields, we need to define the inverse operators $(i\partial^-)^{-1}$ and $(i\partial^+)^{-1}$. We follow the approach used before [2], [3] .

3. The Hamiltonians

We study now the Hamiltonian P^-. With the definitions above the Hamiltonian density $\mathcal{H}$ is

$$\mathcal{H} = \{\psi_+^\dagger \psi_+ e A_P^- + m\psi_+^\dagger \frac{\gamma^-}{4}\psi_- + m\psi_-^\dagger \frac{\gamma^+}{4}\psi_+\} \tag{3.1}$$

where we follow the notation of [12] and observe that the fermionic fields are

$$\psi_+ = \xi \tag{3.2}$$

and

$$\psi_- = \eta \tag{3.3}$$

so that we have

$$\eta = \frac{1}{i\partial^+}\{i\frac{m}{2}\}\xi \tag{3.4}$$

where σ_i are the 2x2 Pauli sigma matrices and now both spinors ξ and η are two-component spinors [12].

Substituting this in the equation [(3.1)] we get

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \tag{3.5}$$

where

$$\mathcal{H}_0 = \frac{1}{2}\xi^\dagger \frac{m^2}{i\partial^+}\xi \tag{3.6}$$

while the interaction Hamiltonian density is equally simple (Notice the factors of $\frac{1}{2}$ in $\mathcal{H}_0$. This turns out to be correct! See [2] for a detailed explanation).

We can write the interaction terms thus:

$$\mathcal{H}_I = \mathcal{H}_{e^+e^-e^+e^-} \tag{3.7}$$

This is as follows:

$$\mathcal{H}_{e^+e^-e^+e^-} = 4e^2\xi^\dagger\xi\frac{1}{(i\partial^+)^2}\xi^\dagger\xi \tag{3.8}$$

Then the generators of x^+ translations are defined to be

$$P_0^- = \frac{1}{2}\int dx^-\mathcal{H}_0 \tag{3.9}$$

$$P_I^- = \frac{1}{2}\int dx^-\mathcal{H}_I \tag{3.10}$$

We obtain the similar forms for P^+ by just interchanging + with – and $_P$ with $_M$, as well as interchanging the spinors ξ with η. For instance, $P^+_{e^-e^+e^+e^-}$ is

$$\mathcal{P}^+_{e^+e^-e^+e^-} = \frac{1}{2}\int dx^+\{4e^2\eta^\dagger\eta\frac{1}{(i\partial^-)^2}\eta^\dagger\eta\} \tag{3.11}$$

4. One-Loop Mass Renormalization

We are ready to apply this to perturbation theory and calculate the fermion mass renormalization and photon mass renormalization to one-loop.

Since both P^+ and P^- are dynamical, we get contributions to the fermion mass from both of them. First, let us compute the contributions from P^-, denoted by δP^-, for an electron with momentum p . We take the approach of using the results of previous paper for $3+1$-dimensions [3] and then taking the limit of $d_t \to 0$ and $m \to 0$. Here $d_t = 2-\epsilon$, the number of transverse dimensions We obtain

$$\delta P^- = \delta P_1^- + \delta P_2^- + \delta P_3^- \tag{4.1}$$

The first contribution becomes, after introducing the parameters $[x] = \frac{[k^+]}{p^+}$ but before taking the limit of $d_t \to 0$ and $m \to 0$:

$$\delta P_1^- = \frac{e^2}{4\pi p^+} \int d(\kappa^2) \int_0^1 dx \frac{1}{[1-x]} \frac{\kappa^2(\frac{4}{[x]^2} - \frac{4}{[x]} + 2)}{x(1-x)p^2 - \kappa^2 - xm^2} \tag{4.2}$$

In the limit of $d_t \to 0$ $m \to 0$ as well as $p^2 \to m^2 = 0$, we get :

$$\delta P_1^- = \tag{4.3}$$

where λ is the infra-red regularization.

The second contribution, δP_2^- becomes

$$\delta P_2^- = \frac{4e^2}{4\pi p^+} \int_0^\infty dx (\frac{1}{[|1-x|^2]} - \frac{1}{[(1+x)^2]}) \tag{4.4}$$

This is no longer zero now, as it was in the $3+1$-dimensional case! Rather, it becomes

$$\delta P_2^- = -\frac{3e^2}{\pi p^+} \tag{4.5}$$

The final contribution, δP_3^-, is

$$\delta P_3^- = \frac{e^2}{2\pi p^+}\int_0^\infty \frac{dx}{[x]}\frac{2}{[1-x]}) \tag{4.6}$$

This becomes

$$\delta P_3^- = \frac{e^2}{2\pi p^+} ln\frac{p^+}{\lambda} \tag{4.7}$$

Here λ is the infra=red regularization.

We compute now easily the contributions from δP^+, by using the explicit parity symmetry of our formalism:

$$\delta P_1^+ = \tag{4.8}$$

with λ as above.

For the other pieces we get $\delta P_2^+ = -\frac{3e^2}{\pi p^-}$ and $\delta P_3^+ = \frac{e^2}{2\pi p^-} ln\frac{p^-}{\lambda}$.

Putting all of this together, we get

$$\delta M^2 = p^+(\delta P^-)_{p^2=0} + p^-(\delta P^+)_{p^2=0}$$

$$= -stuff \tag{4.9}$$

Note two interesting features: first, we have the fact that the infra-red singularities cancel completely; secondly, if we define the mass scale as $\sigma = e/\sqrt{\pi}$, then the fermion mass becomes tachyonic with mass $i\sqrt{\frac{11}{2}}\sigma$. One interpretation is that the fermionic degrees of freedom do not propagate, rather like in the standard calculations, where one computes the vacuum expectation value $<\overline{\psi}\psi>$ and obtains$-\frac{\sigma}{2\pi}e^{\gamma}$, where $\gamma = 0.5772...$ [13] .

We look now at the one-loop quantum correction to the photon mass, for a photon of momentum q. Here we have to be more careful. Since the photon is **not** a physical degree of freedom, we have to resort to some clever way of obtaining the photon measureable quantities. In this paper, we use the idea of taking the $3+1$-dimensional results, and then take the limit of the transverse dimensions and the fermion mass to zero, $d_t \to 0$ $m \to 0$. We also take the momentum q^2 to zero.

The total δP^- is made up of two pieces:

$$\delta P^- = \delta P_1^- + \delta P_2^- \tag{4.10}$$

For teh first contribution, we have

$$\delta P_1^- = (-1)\frac{4e^2}{4\pi q^+}\int d(\kappa^2)\int_0^1 dx \frac{(1-x)xq_\perp^2}{x(1-x)q^2-\kappa^2-m^2} \tag{4.11}$$

Upon taking the limits zero, $d_t \to 0$ $m \to 0$ and m$q^2 \to 0$, we get

$$\delta P_1^- = \frac{e^2}{\pi q^+}\{stuff\} \tag{4.12}$$

where the meaning of λ is as before.

For the second contribution, we have

$$\delta P_2^- = \frac{e^2}{4\pi p^+}\int_0^\infty \frac{dx}{[x]}\frac{2}{[1-x]}) \tag{4.13}$$

This becomes

$$\delta P_2^- = \frac{e^2}{2\pi p^+} ln \frac{p^+}{\lambda} \tag{4.14}$$

We get a similar result for δP^+:

$$\delta P^+ = \frac{e^2}{\pi q^-}\{\frac{5}{4} ln \frac{3}{2} - 1 + \frac{1}{4} ln \frac{q^-}{\lambda}\} \tag{4.15}$$

We obtain similar results for δP^+'s.

The correction to the photon mass becomes then

$$\delta\mu^2 = q^+(\delta P^-)_{q^2=0} + q^-(\delta P^+)_{q^2=0}$$

$$\delta\mu^2 = stuff \tag{4.16}$$

Again, the result is similar to the canonical form, though the coefficients differ: We seem to induce an effective mass σ for the photon (modulo some coefficients which depend on the regularization).

Let us consider now non-pertrubative effects. As we saw in a recent paper [14] , the present quantizaiton give sus the correct number of zero modes in the Schwinger model. This index tells us about the **global**- there were no local contributions - non-perturbative properties of the theory. The number of zero modes for N_F fermions were

$$N_{ZM} = N_{TOP} = \frac{N_F}{2\pi} \int dx^+ dx^- \epsilon_{mn} F^{\mu\nu} \tag{4.17}$$

This gives

$$N_{ZM} == \frac{N_F}{2\pi} \int dx^+ dx^- F^{+-} \tag{4.18}$$

5. Conclusions

Using Penrose quantization, we obtain the usual results for the Schwinger model. The novel results , compared with other light-front quantization schemes, is that we can just use our calculations from the $3 + 1$-dimensional case and take the limit of the vanishing of the perpendicular dimensions and fermion mass, and we obtain the canonical results of non-propagating fermion fields and of inducing an effective photon mass. It is a bit surprising that this limit is so smooth. In naìve light-cone quantization, as done in the discretized approximation, $(DLCQ)$, [15] this limit does not give the correct limit, due to presence of zero modes which are ignored by $DLCQ$. In a sense, the local zero modes are absent in the present treatment. Notice that , as Faddeev and Jackiw point out [10], if the two-forms associated with the change of coordinates from the Lagrangean from to the Hamiltonian form are invertible, then there are no zero modes. In the present formulations, these two forms are Γ eq. (2.20)and Δ, eq. (2.23), both of them invertible. Hence the lack of zero modes [14].

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