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Null Surfaces, Initial Values and Evolution Operators for Spinor Fields⋆

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ABSTRACT

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We analyze the initial value problem for spinor fields obeying the Dirac equation, with particular attention to the characteristic surfaces. The standard Cauchy initial value problem for first order differential equations is to construct a solution function in a neighborhood of space and time from the values of the function on a selected initial value surface. On the characteristic surfaces the solution function may be discontinuous, so the standard Cauchy construction breaks down. For the Dirac equation the characteristic surfaces are null surfaces. An alternative version of the initial value problem may be formulated using null surfaces; the initial value data needed differs from that of the standard Cauchy problem, and in the case we here discuss the values of separate components of the spinor function on an intersecting pair of null surfaces comprise the necessary initial value data. We present an expression for the construction of a solution from null surface data; two analogues of the quantum mechanical Hamiltonian operator determine the evolution of the system.

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1. Introduction

In this work we discuss e spinor fields obeying the free particle Dirac equation, the initial value problem and characteristics.

The standard Cauchy problem for a first order partial differential equation is to use the values of the solution function on an initial value surface to determine the values of the function in a neighborhood of the surface [1]. Solution of the problem proceeds by showing how the initial value data and the differential equation determine all the time derivatives of the function on the initial surface and thus allow a power series development of the function for future times.

If the solution function has a discontinuity across some special surface then the standard Cauchy problem cannot be solved using that surface for the initial value data. Such surfaces are called characteristic surfaces or simply characteristics. The characteristics of the Dirac equation for free particles in flat space are easily shown to be null surfaces. Thus the discontinuities of the solution functions propagate at the velocity of light, independently of the mass parameter in the equations. The characteristics of Maxwell's equations, the Klein Gordon equation, and the vacuum Einstein equations of general relativity are also all null surfaces [2].

We first briefly review the standard Cauchy problem for the Dirac equation. We then obtain the characteristics and solve the analogous problem for the two planar null surfaces $t = /pmx$; we suppress dependence on y, z coordinates. Initial data for this situation consists of the values of disparate components of the spinor function on the pair of null surfaces. We give a simple evolution operator expression in terms of null coordinates, which is a direct analogue of the usual expression of quantum theory, except that two analogues of the Hamiltonian operator determine the evolution of the solution, that is are generators of displacements in the null coordinates [3, 4, 5, 6, 7].

2. Development Of a Spinor Function From a Constant Time Hypersurface

Since the Dirac equation may be expressed in Hamiltonian form the development of a solution from initial value data given on a constant time hypersurface is one of the most familiar problems of physics [8], and we give here a very brief reminder. We express the Dirac equation in Hamiltonian form as [9]

$$i\psi_{|t}(\vec{x},t) = [-i\vec{\alpha}.\bigtriangledown + \beta m]\psi(\vec{x},t) = H\psi(\vec{x},t) \tag{2.1}$$

(We use units in which $\hbar = c = 1$. The slash notation indicates differentiation with respect to the indicated variable, in this case time t.) We write a Taylor series expansion as

$$\psi(\vec{x},t) = \psi(\vec{x},t)_{t=0} + t\psi_{|t}(\vec{x},t)_{t=0} + \frac{t^2}{2!}\psi_{|t|t}(\vec{x},t)_{t=0} + ... \tag{2.2}$$

All of the time derivatives at $t = 0$ may be readily obtained from the Dirac equation and the value of the spinor function at $t = 0$, which we call $h(\vec{x})$. When substituted in the series these give the familiar result

$$\psi(\vec{x},t) = \psi(\vec{x},t)_{t=0} + (-iHt)\psi(\vec{x},t)_{t=0} + \frac{(-iHt)^2}{2!}\psi(\vec{x},t)_{t=0} + ... = e^{-iHt}h(\vec{x}) \tag{2.3}$$

The indicated exponential of the Hamiltonian operator is thereby seen as the finite time displacement operator.

3. Characteristics of the Dirac Equation

The characteristics of the Dirac equation are found in a standard way [1], and, since the Dirac equation is first order in time, the problem is even easier. We suppose that the initial value of the solution is given on a hypersurface S, $t = T(\vec{x})$, by a four component spinor

$$\psi(\vec{x}, T(\vec{x})) = h(\vec{x}) \tag{3.1}$$

and that the solution is given by a Taylor series expansion

$$\psi(\vec{x}, t) = \psi(\vec{x}, t)_{t=T} + [t - T(\vec{x})]\psi_{|t}(\vec{x}, t)_{t=T} + \frac{[t - T(\vec{x})]^2}{2!}\psi_{|t|t}(\vec{x}, t)_{t=T} + ... \tag{3.2}$$

To find the first time derivative in (3.2) we first express the gradient of the initial value h in terms of $t = T(\vec{x})$, as

$$\bigtriangledown h(\vec{x}) = \bigtriangledown \psi(\vec{x}, t)_{t=T} + \psi_{|t}(\vec{x}, t)_{t=T} \bigtriangledown T(\vec{x}) \tag{3.3}$$

From this and the Dirac equation we may write an equation for the time derivative of ψ on S,

$$i[I - \vec{\alpha}. \bigtriangledown T(\vec{x})]\psi_{|t}(\vec{x}, t)_{t=T} - -i\vec{\alpha}. \bigtriangledown h(\vec{x}) + \beta m h(\vec{x}) \tag{3.4}$$

This may be solved for the time derivative if the matrix in square brackets has an inverse. The determinant of that matrix is easily found to be

$$|I - \vec{\alpha}.\nabla T(\vec{x})| = (1 - \nabla T(\vec{x})^2)^2 \tag{3.5}$$

Thus the condition that S be a characteristic is that the above quantity be zero, or

$$\nabla T(\vec{x})^2 = 1 \tag{3.6}$$

This is the same as the characteristic equation for the Maxwell's equation [1], [2]; thus the characteristics are null surfaces, and are independent of the mass parameter in the Dirac equation. The particular null surfaces that we will study in the remainder of this paper are

$$u = t - x = 0 \quad u \quad characteristic \quad or \quad null \quad surface$$

$$v = t + x = 0 \quad v \quad characteristic \quad or \quad null \quad surface \tag{3.7}$$

4. Construction of a Solution Function in Terms of Null Coordinates

The covariant form of the Dirac equation is [9]

$$\gamma^\alpha i \frac{\partial}{\partial x^\alpha} \psi = m\psi \tag{4.1}$$

We will study this using the null coordinates $u = t - x$ and $v = t + x$, and suppress dependence on y and z in all that follows [10]. Then the gamma matrices are $\gamma^u = \gamma^0 - \gamma^1$ and $\gamma^v = \gamma^0 + \gamma^1$. The Dirac equation and the gamma matrix algebra in terms of the null coordinates are

$$[\gamma^u i \frac{\partial}{\partial u} + \gamma^v i \frac{\partial}{\partial v}] \psi(u, v) = m\psi(u, v) \tag{4.2}$$

$$\frac{\gamma^u \gamma^v}{4} + \frac{\gamma^v \gamma^u}{4} = 1, \quad (\gamma^u)^2 = (\gamma^v)^2 = 0 \tag{4.3}$$

We define a pair of projection operators as

$$\Lambda_u = \frac{\gamma^u \gamma^v}{4} \quad \Lambda_v = \frac{\gamma^v \gamma^u}{4} \tag{4.4}$$

From (4.3)the usual projection operator properties follow, that is

$$\Lambda_u + \Lambda_v = 1, \quad \Lambda_u \Lambda_u = \Lambda_u, \quad \Lambda_v \Lambda_v = \Lambda_v, \quad \Lambda_u \Lambda_v = \Lambda_v \Lambda_u = 0 \tag{4.5}$$

To analyze the Dirac equation on and near the null surfaces $u = 0$ and $v = 0$ we make an expansion in the mass m, since the solutions of the zero mass Dirac equation are functions of only u or only v. Thus we write

$$\psi(u, v) = \psi^{(0)}(u, v) + m\psi^{(1)}(u, v) + m^2\psi^{(2)}(u, v) + ... \tag{4.6}$$

Substitution of this into the Dirac equation gives the following set of iterative equations

$$[\gamma^u i \frac{\partial}{\partial u} + \gamma^v i \frac{\partial}{\partial v}]\psi^{(0)}(u, v) = 0, \quad [\gamma^u i \frac{\partial}{\partial u} + \gamma^v i \frac{\partial}{\partial v}]\psi^{(n)}(u, v) = m\psi^{(n-1)}(u, v) \tag{4.7}$$

Solution of the zero order equation is easy by inspection,

$$\psi^{(0)}(u, v) = \Lambda_u f(u) + \Lambda_v g(v) \tag{4.8}$$

Here the functions f and g are any differentiable 4-tuple functions of u and v; we will call these the generating functions. It is instructive to consider a representation of the gamma matrices in which the projection operators Λ are diagonal:

$$\Lambda_u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

$$\Lambda_v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

As for the zero order solution , they are:

$$\psi^{(0)}(u,v) = \begin{pmatrix} f_1(u) \\ f_2(u) \\ g_1(v) \\ g_2(v) \end{pmatrix} \tag{4.11}$$

Thus only two components of f and two components of g enter in the solution. The first order equation (4.7) is

$$[\gamma^u i\frac{\partial}{\partial u} + \gamma^v i\frac{\partial}{\partial v}]\psi^{(1)}(u,v) = \Lambda_u f(u) + \Lambda_v g(v) \tag{4.12}$$

Only a particular solution to this is needed since the homogeneous solution may be absorbed into the zero order solution. We break the equation into u and v parts by premultiplying by Γ_u and using (4.2) to get

$$i\frac{\partial}{\partial v}\Lambda_u \psi^{(1)}(u,v) = \frac{1}{4}\gamma^u \Lambda_v g(v) \tag{4.13}$$

The right side of this is a function of v only, so the left side must be also. Thus we may write a solution by integration.

$$\Lambda_u \psi^{(1)}(u,v) = \frac{-i}{4}\gamma^u \Lambda_v \int\limits_{v_0}^{v} dv' g(v') \tag{4.14}$$

Here v_0 is an arbitrary parameter. In the same way we may obtain a similar expression for the v projection of the solution,

$$\Lambda_v \psi^{(1)}(u,v) = \frac{-i}{4}\gamma^v \Lambda_u \int\limits_{u_0}^{u} du' f(u') \tag{4.15}$$

The sum of (4.14) and (4.15) gives the complete first order solution

$$\psi^{(1)}(u,v) = \frac{-i}{4}[\gamma^u \Lambda_v \int\limits_{v_0}^{v} dv' g(v') + \gamma^v \Lambda_u \int\limits_{u_0}^{u} du' f(u')] \tag{4.16}$$

The second order equation (4.7) may now be written as

$$i\frac{\partial}{\partial v}\Lambda_u \psi^{(2)}(u,v) = \frac{-i}{4}[\gamma^u \Lambda_v \int\limits_{v_0}^{v} dv' g(v') + \gamma^v \Lambda_u \int\limits_{u_0}^{u} du' f(u')] \tag{4.17}$$

As before we premultiply by γ^u to obtain, with use of (4.4) and (4.5),

$$\frac{\partial}{\partial v}\Lambda_u \psi^{(2)}(u,v) = \frac{-1}{4}\Lambda_u \int\limits_{u_0}^{u} du' f(u') \tag{4.18}$$

Since the right side of this is a function of only u we may write a particular solution by inspection,

$$\Lambda_u \psi^{(2)}(u,v) = \frac{-1}{4}(v-v_0)\Lambda_u \int\limits_{u_0}^{u} du' f(u') \tag{4.19}$$

In the same way we may obtain the v projection of the solution,

$$\Lambda_v \psi^{(2)}(u,v) = \frac{-1}{4}(u-u_0)\Lambda_v \int_{v_0}^{v} dv' g(v') \qquad (4.20)$$

Summing (4.19)and (4.20)we have the complete second order solution,

$$\psi^{(2)}(u,v) = \frac{-1}{4}(v-v_0)[\Lambda_u \int_{u_0}^{u} du' f(u') + (u-u_0)\Lambda_v \int_{v_0}^{v} dv' g(v')] \qquad (4.21)$$

The procedure is now clear, and we may continue to all orders; the even orders are similar in form, and the odd orders are similar in form. The complete series solution is

$$\psi(u,v) = \sum_{n=0}^{\infty} \frac{[-\frac{m^2}{4}(v-v_0)\Gamma_u]^n}{n!}(1-\frac{im}{4}\gamma^v \Gamma_u)\Lambda_u f(u) + \sum_{j=0}^{\infty} \frac{[-\frac{m^2}{4}(u-u_0)\Gamma_v]^j}{j!}(1-\frac{im}{4}\gamma^u \Gamma_v)\Lambda_v g(v) \qquad (4.22)$$

where the multiple integral operator Γ_u^n is defined as

$$\Gamma_u^n f(u) = \int_{u_0}^{u} \int_{u_0}^{u'} \ldots \int_{u_0}^{u^{n-1}} du^n f(u^n) \qquad (4.23)$$

The series (29) may be readily summed to give a concise expression for the solution

$$\psi(u,v) = e^{-\frac{m^2}{4}(v-v_0)\Gamma_u}(1-\frac{im}{4}\gamma^v \Gamma_u)\Lambda_u f(u) + e^{-\frac{m^2}{4}(u-u_0)\Gamma_v}(1-\frac{im}{4}\gamma^u \Gamma_v)\Lambda_v g(v) \qquad (4.24)$$

This gives a solution of the free particle Dirac equation for any pair of generating functions f and g; the generating functions together have only four independent components however, so this initial value data contains the same amount of information as that in the standard Cauchy problem.

5. Initial Values on Null Surfaces

The generating functions f and g are arbitrary functions, and are simply related to the initial values of the solution on the null surfaces as we will now discuss. In this and the following section we set $u_0 = v_0 = 0$ without loss of generality.

To get a relation between the initial values of the function y on the null surfaces and the generating functions f and g we set $u = 0$ and then $v = 0$ in (4.24) and easily find

$$\psi_0(u,0) = (1 - \frac{im}{4}\gamma^v\Gamma_u)\Lambda_u f(u) + \Gamma_v\Lambda_v g(0) \tag{5.1}$$

$$\psi_0(0,v) = (1 - \frac{im}{4}\gamma^u\Gamma_v)\Lambda_v g(v) + \Gamma_u\Lambda_u f(0) \tag{5.2}$$

We wish to make a convenient choice for the values of $f(0)$ and $g(0)$; taking $u = v = 0$ in (5.1), (5.2) we see that

$$\psi_0(0,0) = \Lambda_u f(0) + \Lambda_v g(0) \tag{5.3}$$

Accordingly we choose $f(0) = g(0) = \psi_0(0,0)$; then the quantities that appear in the evolution expression (4.24) are

$$(1 - \frac{im}{4}\gamma^v\Gamma_u)\Lambda_u f(u) = \psi_0(0,0) - \Lambda_v\psi_0(0,0)) \tag{5.4}$$

$$(1 - \frac{im}{4}\gamma^u\Gamma_v)\Lambda_v g(v) = \psi_0(0,0) - \Lambda_u\psi_0(0,0)) \tag{5.5}$$

Thus the spinorial expressions in parentheses turn the arbitrary 4-tuple functions into the indicated initial data on the null surfaces. These expressions may be inverted if it is desired to obtain f and g.

$$\Lambda_u f(u) = (1 + \frac{im}{4}\gamma^v \Gamma_u)[\psi_0(0,0) - \Lambda_v \psi_0(0,0)] \tag{5.6}$$

$$\Lambda_v g(v) = (1 + \frac{im}{4}\gamma^u \Gamma_v)[\psi_0(0,0) - \Lambda_u \psi_0(0,0)] \tag{5.7}$$

In terms of the initial values of the solution the expression (31) now reads

$$\psi(u,v) = e^{-\frac{m^2}{4}(v-v_0)\Gamma_u}(\psi_0(0,0) - \Lambda_v \psi_0(0.0)) + e^{-\frac{m^2}{4}(u-u_0)\Gamma_v}(\psi_0(0,0) - \Lambda_u \psi_0(0,0)) \tag{5.8}$$

This is a complete expression for the evolution of the solution from its values on the pair of null surfaces $u = 0$ and $v = 0$. Note that only the square of the mass appears in the evolution operators.

6. Plane Waves

We wish to verify the consistency of (5.8) for plane wave solutions of the Dirac equation. We will show that if the appropriate initial data functions are put into (5.8) then the solution function generated is the appropriate plane wave. We write a plane wave solution in Cartesian and null coordinates as

$$\psi(u,v) = e^{-iEt+ikx} w(E,k) = e^{-i\lambda u - i\tau v} w(\lambda, \tau) \tag{6.1}$$

Here w is the usual Dirac 4-tuple spin function and the null momenta are given by

$$\lambda = \frac{E+k}{2}, \tau = \frac{E-k}{2}, \lambda\tau = \frac{E^2-k^2}{4} = \frac{m^2}{4} = \mu^2 \tag{6.2}$$

Thus the appropriate initial value quantities that enter the evolution expression

(5.8)are

$$\psi_0(u,0) - \Lambda_v \psi_0(0,0) = (e^{-i\lambda u} - \Lambda_v) w(\lambda, \tau)$$

$$\psi_0(0,v) - \Lambda_u \psi_0(0,0) = (e^{-i\tau v} - \Lambda_u) w(\lambda, \tau) \tag{6.3}$$

We substitute these expressions into (5.8), abbreviating for convenience $B = -i\lambda, C = -i\tau, -m^2/4 = BC$, and find

$$\psi(u,v) = (e^{BCv\Gamma_u}(e^{Bu} - \Lambda_v) w(\lambda, \tau) + (e^{BCu\Gamma_v}(e^{Cv} - \Lambda_u) w(\lambda, \tau)$$

$$= (e^{BCv\Gamma_u} e^{Bu} + e^{BCu\Gamma_v} e^{Cv}) w(\lambda, \tau) - F_c(uv) w(\lambda, \tau) \tag{6.4}$$

Here F_c denotes the function related to a constant generator, defined and evaluated as

$$F_c(uv) = \sum_{n=0}^{\infty} \frac{(-\mu^2 uv)^2}{(n!)^2} = e^{-\frac{m^2}{4} v\Gamma_u} 1 = e^{-\frac{m^2}{4} u\Gamma_v} 1 \tag{6.5}$$

Notice that this function is symmetric in u and v. The quantity in parentheses in (6.4) is straight forward to evaluate; we expand the exponentials in double series as

$$(e^{BCv\Gamma_u} e^{Bu} + e^{BCu\Gamma_v} e^{Cv}) = \sum_{n=0,j=0}^{\infty} \frac{(BCv\Gamma_u)^n}{n!} \frac{(Bu)^j}{j!} + \sum_{m=0,k=0}^{\infty} \frac{(BCu\Gamma_v)^m}{m!} \frac{(Cv)^k}{k!} \tag{6.6}$$

It is easy to obtain the operation of Γ^n on powers of u,

$$\Gamma_u^n u^j = \frac{u^{j+n}}{(j+1)(j+2)...(j+n)} \tag{6.7}$$

We substitute this into (6.6) and rearrange summation indices to find

$$e^{BCv\Gamma_u}e^{Bu}+e^{BCu\Gamma_v}e^{Cv})=\sum_{n\le j}^{\infty}\frac{(Cv)^n}{n!}\frac{(Bu)^j}{j!}+\sum_{m\ge k}^{\infty}\frac{(Cv)^m}{m!}\frac{(Bu)^k}{k!}$$

$$=\sum_{n=0,j=0}^{\infty}\frac{(Cv)^n}{n!}\frac{(Bu)^j}{j!}+\sum_{m=0,k=0}^{\infty}\frac{(Cv)^m}{m!}\frac{(Bu)^k}{k!}+\sum_{n=0}^{\infty}\frac{(-\mu^2uv)^2}{(n!)^2}=e^{Bu+Cv}+F_c(uv) \tag{6.8}$$

We now combine (6.3) and (6.7) to get the complete solution

$$\psi(u,v)=e^{Bu+Cv}w(\lambda,\tau)=e^{-i\lambda u-i\tau v}w(\lambda,\tau) \tag{6.9}$$

which is the correct plane wave solution. We have thus checked the consistency of a known solution with our formalism. Moreover it follows that the evolution operator equation (5.8) will produce the correct solution in any case that may be expanded as a superposition of plane waves.

We note that the constant terms in the initial data expression (5.4), (5.5) play a very important role in the formalism and cannot be ignored.

7. Relation to the Hamiltonian Viewpoint

The evolution operators in (5.8) are close symbolic analogues of the usual evolution operator in expression (2.3), as we will discuss in some detail.

A Taylor series expansion in t of a function may be expressed symbolically as an exponential as follows

$$\psi(\vec{x},t)=e^{(t-t_0)\frac{\partial}{\partial t}}\psi(\vec{x},t_0)\equiv\sum_{n=0}^{\infty}\frac{(t-t_0)^n}{n!}\psi_{|nt}(\vec{x},t)_{t=t_0} \tag{7.1}$$

That is the quantity in the exponent generates a time shift. In standard quantum mechanics the time derivative is related to the energy operator by $\frac{\partial}{\partial t}=-iE$. To define the dynamics the energy operator in terms of t is identified with the Hamiltonian operator H in terms of x. That is a Schroedinger equation is postulated, with an operator equivalence

$$\frac{\partial}{\partial t} = -iH \tag{7.2}$$

Then for a wave function the Taylor series expression reads

$$\psi(\vec{x}, t) = e^{-i(t-t_0)H}\psi(\vec{x}, t_0) \tag{7.3}$$

which is the standard form for the time evolution operator, as in (2.3).

We can use the same procedure to obtain our result (5.8)- but only symbolically and up to a normalization and constant terms. We begin with an expression for a double Taylor series expansion of a function of u and v in analogy with (7.1).

$$\psi(u, v) = \frac{1}{2}[e^{(v-v_0)\frac{\partial}{\partial v}}\psi(u, v_0) + e^{(u-u_0)\frac{\partial}{\partial u}}\psi(u_0, v)] \tag{7.4}$$

In analogy with the Schroedinger equation we define the dynamics with the Klein Gordon equation, which in terms of u and v is,

$$\frac{\partial^2}{\partial u \partial v}\psi(u, v) = -\frac{m^2}{4}\psi(u, v) \tag{7.5}$$

- which solutions of the Dirac equation must obey. In analogy with (7.2) this allows us to identify the v derivative operator with the following operator in terms of u,

$$\frac{\partial}{\partial v} = -\frac{m^2}{4}(\frac{\partial}{\partial u})^{-1} \equiv -\frac{m^2}{4}\Gamma_u \tag{7.6}$$

Substituting this into (7.4) we obtain (5.8)- if we do not concern ourselves about the normalization factor and the constant terms, which the symbolic derivation does not seem to explain.

In conclusion our main result (5.8) presents the evolution of a spinor field in terms of two analogues of the Hamiltonian operator.

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