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When are Zero Modes Necessary ?[⋆]

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ABSTRACT

In this paper we develop a way to count zero momentum modes in front-form quantization and apply it to several such approaches: canonical light-cone quantization; Dirac light-cone quantization; reduced phase space quantization ; and Penrose quantization.

We also examine which of these light-cone methods are consistent with the Atiyah-Singer index theorem.

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1. Introduction

In recent years there has been a lot of interest in zero modes in light-cone quantized field theories [1] [2] [3] [4] . Yet it is not always clear just **when** are they needed. In this paper we will attempt to answer this question. We will first write down an expression which counts the total number of zero modes. We then apply this formula to several recent approaches to light-front quantization : canonical light-cone quantization [5] in which one quantizes the fields along the surface $x^+ = 0$, in close analogy to the canonical equal-time quantization; [6] Dirac light-cone quantization [7] where one applies the ideas introduced by Dirac, [8] which tell us that in light-cone coordinates one deals with a system with constrains, so that the canonical quantization procedure is not sufficient, and new brackets, Dirac brackets need to be defined; reduced phase space quantization [9] [10] [11] where we use a geometrical method of identifying the independent degrees of freedom, and quantize only them; and Penrose quantization [12] [13] [14] [15] [16] [17] which is this new method I have been developing lately, in which we initialize the fields on the surfaces $x^+ = 0$, $x^- = 0$ as well as $x^+ = x^- = 0$.

We will then compare these different methods in their predictions for this zero mode index N_{ZM} and also see how it compares with predictions of the Atiyah-Singer index theorem [18] in the context of several theories: pure scalar ϕ^4 ; fermion-boson Yukawa coupling, and QED_d , for $d = 1 + 1$ and $d = 3 + 1$.

2. The Zero Mode Index

Our starting point will be a formula developed by E. Abdalla, M. C. B. Abdalla and K. D. Rothe [19] which relates the index of operators with the number of zero modes N_{ZM}

$$N_{ZM} = N_G + N_L \tag{2.1}$$

where N_G are the contributions to the number of zero modes due to global factors, and N_L due to local factors. In their book, Abdalla , Abdalla and Rothe apply this to a particular form of two-dimensional theories where

$$N_G = \frac{a_{\frac{d}{m}}(x)}{(4\pi)^{\frac{d}{m}}} \tag{2.2}$$

and

$$N_L = -\zeta(0; x, x|D) \tag{2.3}$$

$a_{\frac{d}{m}}$ are Seely coefficients used to compute the global operators contributing to the number of zero modes, where $d = 2$ and $m = 1$, while ζ is a generalized zeta-function used to compute the trace of the local operator $i\not{D}$.

What we propose to do now is to generalize this formula so that we can apply it to our more general case. The generalization we propose is to have more contributions to both the global and local number of zero modes, by adding what we have learned from light-cone field theory [20] as well as from supestring theory [21] .

In particular, the global zero modes will be like this

$$N_G = N_{TOP} + N_{GJ} + N_{BT} \tag{2.4}$$

where N_{TOP} are contributions from topological terms; N_{GJ} are contributions from zeros in the Jacobian of global transformations; and N_{BT} are zero modes from global boundary terms.

In a similar fashion , the local contributions are now

$$N_L = -N_{i\not{D}} + N_{LJ} \tag{2.5}$$

where $-N_{i\not{D}}$ is the contribution from the Dirac operator, and N_{LJ} the contribution from the zeros of the Jacobian of local conformal transformations.

Let us describe each of these terms. N_{TOP} is the contribution to the global number of zero modes due to topological features of the theory. In the presence of a gauge symmetry, this term is the contributions due to the induced two-form $F^{\mu\nu}$. In 1 + 1-dimensions, this is (for N_F fermions)

$$N_{TOP} = \frac{N_F}{2\pi}\int d^2x\epsilon_{\mu\nu}F^{\mu\nu} = \frac{a_{\frac{d}{m}}(x)}{(4\pi)^{\frac{d}{m}}} \tag{2.6}$$

The next term is N_{GJ}. This is the term which contributes to the global number of zero modes from the zeros in the Jacobian of global transformations. In light-cone quantization, we map the coordinates $(q, \frac{dq}{dt})$ to (q, p). It turns out that in many cases, this mapping is singular in that the Jacobian vanishes [22] [23] [24] [25] [26] . In that case, we have a zero mode from each of the zeros of this determinant. Just like in the N_{TOP} we get zero modes from the induced two-form $F^{\mu\nu}$ so here we may get zero modes from the simplectic two-form f^{ij} [9].

Finally, the last contribution to global zero modes comes from boundary terms. In front-form, the fields can have non-trivial behaviour at ∞, which shows that we can't ignore these terms as is costumary in equal-time quantization [20] [27] . In most cases, this term does not contribute, since it contains terms proportional to the momentum of the fields, which falls off faster at ∞ than the field itself, hence can be dropped. In the case of fermion fields, though, the momentum conjugate to ψ is proportional to $\psi^\dagger$, so these terms can't be ignored - this was essentially the finding of Langnau [20] for the fermion propagator in perturbation theory.

We examine now the local contributions. Firstly, we have $-N_{i\not{D}}$. The reason for the sign can be seen in looking at the generating functional Z [28] [29] and noticing that functional integration gives a factor of $Det(i\not{D})$ for the fermion degrees of freedom, but a factor of $Det(-D^2)^{-\frac{1}{2}}$ for each bosonic degree of freedom. Since the Trace of the operator involves $Ln(Det(Z))$, this gives different signs for fermionic and bosonic degrees for freedom.

Finally, the term N_{LJ} measures the contribution from the zeros of the Jacobian for local (conformal) transformation [21] . In most of the applications we will consider, this term will not contribute since our transformation to light-cone coordinates is non-singular locally (conformally).

We will study now how these terms are realized in the four cases mentioned above. Note that there might be cancelations due to the fact the fermions contribute with opposite sign compared with the bosons.

3. The Zero Mode Index for Canonical Light-Cone Quantization

Let us study first the pure scalar theory. Since there are no induced two-forms there are no contributions from N_{TOP}. In this method the zero modes from N_{GJ} are ignored, so there is no contribution from this direction. Also, the boundary terms are taken to vanish sufficiently fast at ∞ so that N_{BT} also does not contribute.

Going to the local terms, we see that does not N_{LJ} contribute, so that we get

$$N_{ZM} = 0 \tag{3.1}$$

Let us look now at the Yukawa case. The difference with the pure scalar case is the presence of fermions. In the absence of any internal symmetry [30] there are no induced two-form so N_{TOP} is still zero . The boundary terms might contribute, though they are assumed to vanish fast enough in this formalism. The only terms which are potentially contributing are $-N_{i\not{D}}$. Let us look at these operators, since their form depends on which are the independent degrees of freedom. In the canonical case, only ψ_+ and $\psi_+^\dagger$ are independent, while ψ_- and $\psi_-^\dagger$ are dependent. In this case, $Det(i\not{D})$ is obtained from functional integration over the fermionic degrees of freedom. It is

$$Det(i\not{D}) = Det\{\frac{\partial^+}{2}1_{2\times 2} + [\sigma_i\frac{i\partial_i}{2} + V(\phi)1_{2\times 2}]\frac{1}{i\partial^+}[\sigma_i\frac{i\partial_i}{2} + V(\phi)1_{2\times 2}]\} \tag{3.2}$$

where σ are the 2×2 Pauli matrices and $1_{2\times 2}$ is the 2×2 identity matrix, and where $V(\phi)$ is a polynomial in ϕ. In the limit of $1+1$-dimensions, difficulties can come from either zero modes of the field ϕ itself or from the zero modes of the operator ∂^+. But both of these possibilities are repressed by the present formalism, in that these zero modes are disallowed. Therefore,

$$N_{ZM} = 0 \tag{3.3}$$

We come then to QED. It is known that here we have contributions from the topological term. In $1+1$-dimensions, it is

$$N_{TOP} = \frac{N_F}{2\pi}\int d^2x \epsilon_{\mu\nu} F^{\mu\nu} \tag{3.4}$$

In the light-cone gauge, $A^+ = 0$, this becomes

$$N_{TOP} = \frac{N_F}{2\pi}\int dx^+ [A^-(x^+,\infty) - A^-(x^+,-\infty)] \tag{3.5}$$

which is usually zero unless we specify particular values at ∞ for the longitudinal coordinate [11] . So this is zero, which is not the general result. It is for this reason that McCartor introduced his construction [31] . In $3+1$-dimensions, we have a similar integral

$$N_{TOP} = \frac{N_F}{16\pi^2}\int d^2x_\perp \frac{dx^+ dx^-}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{3.6}$$

which becomes

$$N_{TOP} = \frac{N_F}{16\pi^2}\int dx^2 x_\perp dx^+ \{\epsilon_{ij}(\partial^i A^j - \partial^j A^i)[A^-(x^+,x_\perp,\infty) - A^-(x^+,x_\perp,-\infty)]$$

$$+\epsilon_{ij}(\partial^j A^- - \partial^- A^j)[A^i(x^+,x_\perp,\infty) - A^i(x^+,x_\perp,-\infty)]\} \tag{3.7}$$

Again, with usual boundary conditions, this is zero, while we know that it should be some general integer.

We also might have contributions from the boundary terms, also set forcibly to zero. As for $-N_{i\not D}$, N_{JL}, they are both zero as before, so we obtain

$$N_{ZM} = 0 \tag{3.8}$$

So in the canonical approach, we never have zero modes, even in the case of QED when we should !

4. The Zero Mode Index for Dirac Light-Cone Quantization

We start again with the pure scalar theory. In this method, we **do** take into account the zero modes from N_{GJ}, which are found to contribute [1, 7]. We get one unit from each scalar field present, so for N_S scalars we have

$$N_{GJ} = N_S \tag{4.1}$$

The boundary terms are also ignored in this approach, so it is zero. As for the local terms, again, N_{LJ} does not contribute, so we obtain

$$N_{ZM} = N_S \tag{4.2}$$

hence here we do have to worry about zero modes, even for the scalar theory!

We go now to the Yukawa case. We still have no induced two-form , so N_{TOP} is still zero. The presence of fermions now brings an extra contribution for each N_F fermions to N_{GJ}, so that it is

$$N_{GJ} = N_S + N_F \tag{4.3}$$

We also have to worry now about the fermionic contribution, $-N_{i\not{\partial}}$. Here the independent degrees of freedom are also only ψ_+ and $\psi_+^\dagger$. The $Det(i\not{D})$ is now

$$Det(i\not{D}) = Det\{\frac{\partial^+}{2}1_{2\times2}+[\sigma_i\frac{i\partial_i}{2}+V(\omega+\varphi)1_{2\times2}]\frac{1}{i\partial^+}[\sigma_i\frac{i\partial_i}{2}+V(\omega+\varphi)1_{2\times2}]\} \tag{4.4}$$

where ω is the zero mode of the field ϕ and φ is the remaining non-zero mode piece [1, 7]. Even in the massless case, the zero mode ω 'regulates' this determinant (since it is a 'c-number'), so that the fermionic determinant itself has not zero modes . We still have then

$$N_{ZM} = N_S + N_F \tag{4.5}$$

We finally look at QED. Here, there **is** a topological contribution, and this was found to be given correctly in this formalism [7] . Plus, we still have a contribution from N_{GJ} from the zero modes of the gauge field and the fermion field, so we have

$$N_{GJ} = N_S + N_F \tag{4.6}$$

The local terms still don't contribute, the gauge zero modes still shielding the fermionic determinant. The final result is

$$N_{ZM} = N_{TOP} + N_S + N_F \tag{4.7}$$

where , in $1+1$-dimensions

$$N_{TOP} = \frac{N_F}{2\pi} \int d^2x \epsilon_{\mu\nu} F^{\mu\nu} \tag{4.8}$$

while in $3+1$-dimensions this is

$$N_{TOP} = \frac{N_F}{16\pi^2} \int d^2x_\perp \frac{dx^+ dx^-}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{4.9}$$

and we can show that these are integers at least for the $1+1$ case [7] .

In conclusion, in the case of Dirac quantization, we **always** need to worry about zero modes, since our index was always non-zero, even for the rather simple scalar theories.

5. The Zero Mode Index for Reduced Phase-Space Light-Cone Quantization

We examine now the reduced phase-space method, and we start with pure scalar case. There is no topological contribution, no global Jacobian contribution - the simplectic two-form seems invertible here [9] - and there are no boundary term contributions. So

$$N_G = 0 \tag{5.1}$$

Since there are no fermions and the local Jacobian also does not contribute, we obtain

$$N_{ZM} = 0 \tag{5.2}$$

We look now at Yukawa theory. The inclusion of fermions changes things a bit. We still have no global Jacobian contributions, but might have contributions from the boundary term and from $-N_{i\not{\partial}}$. With the choice of anti-periodic boundary conditions at longitudinal ∞, we get no contribution from the boundary term since the fermionic piece, the only one which might contribute, is quadratic in the field

$$N_{BT} = \int dx^2 dx^- \psi_+^\dagger \psi_= |_{x^+=-\infty}^{x^+=+\infty} \tag{5.3}$$

so that it cancels! Hence, $N_{BT} = 0$. For the fermionic term we get

$$Det(i\not{D}) = Det\{\frac{i\partial^+}{2} 1_{2\times 2} + [\sigma_i \frac{i\partial_i}{2} + V(\phi) 1_{2\times 2}] \frac{1}{i\partial^+} [\sigma_i \frac{i\partial_i}{2} + V(\phi) 1_{2\times 2}]\} \tag{5.4}$$

Now the prescription of taking anti-periodic values at longitudinal ∞ demands that $(i\partial^+)^{-1}$ be anti-periodic, hence without zero modes. In view of these points, there are no zero modes from eq. [(5.4)]. Therefore,

$$N_{ZM} = 0 \tag{5.5}$$

Finally, we look at QED. Here, there are topological contributions, which can be shown to give the right value as shown above in eqs. [(4.8), (4.9)]. We have as follows for $1+1$-dimensions

$$N_{ZM} = N_{TOP} = \frac{N_F}{2\pi} \int d^2 x \epsilon_{\mu\nu} F^{\mu\nu} \tag{5.6}$$

while in $3+1$-dimensions we have

$$N_{ZM} = N_{TOP} = \frac{N_F}{16\pi^2} \int d^2 x_\perp \frac{dx^+ dx^-}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{5.7}$$

Therefore, in the case of reduced phase-space quantization, we need to worry about zero modes only in the case when there is an induced two-form with non-zero value, as expected. The scalar or fermionic degrees of freedom do not contribute.

6. The Zero Mode Index for Penrose Quantization

We come now to the Penrose quantization. In many respects, this method shares the features of the reduced phase-space quantization. One important different is that here both ψ_+ and ψ_- are taken as independent.

For the pure scalar case, it is clear that

$$N_{ZM} = 0 \tag{6.1}$$

since none of the terms contribute, like in the reduced phase space quantization method discussed above.

For the Yukawa case, we saw [16] that the simplectic two-forms f^{ij} (called Δ and Γ in that paper) are both invertible, so no zero modes from that place. As for the fermions we take here both ψ_+ and ψ_- as independent variables, so that the analysis is a bit more complicated. The boundary terms still vanish, though they look like this

$$N_{BT} = \int dx^2 dx^- \psi_+^\dagger \psi_+|_{x^+=-\infty}^{x^+=+\infty} + \int dx^2 dx^+ \psi_-^\dagger \psi_-|_{x^-=-\infty}^{x^-=+\infty} \tag{6.2}$$

As before, this still vanishes, so that $N_{BT} = 0$ as above, eq. [(5.4)]. We might have a contribution from $-N_{i\not{\partial}}$.

$$Det(i\not{D}) = -\frac{1}{4}(\partial^+\partial^- - \partial_i\partial^i) + V(\phi)^2 \tag{6.3}$$

We can see from this form that there are no zero modes contributing,

$$N_{ZM} = 0 \tag{6.4}$$

Then, to QED. We get the usual topological terms, [(4.8), (4.9)], which display the expected behaviour. We also know from other calculations , at least in the $1+1$-dimensions [13], that the anomaly works out in Penrose quantization

$$N_{ZM} = N_{TOP} = \frac{N_F}{2\pi} \int d^2x \epsilon_{\mu\nu} F^{\mu\nu} \tag{6.5}$$

while in $3+1$-dimensions this is

$$N_{ZM} = N_{TOP} = \frac{N_F}{16\pi^2} \int d^2x_\perp \frac{dx^+ dx^-}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{6.6}$$

In conclusion, in Penrose quantization, like in reduced phase-space quantization, we need zero modes only when we have induced two-forms due to the presence of gauge fields or a gauge symmetry [30] . Furthermore, as a general case, it seems that we don't encounter contributions from the global Jacobian, so that the simplectic forms seem always invertible, hence there are **no** constrains, as originally pointed out by Penrose [12] .

7. Conclusions

We have examined four methods of light-cone quantization in light of how they treat zero modes, by employing a new index which counts them (eqs. [(2.1), (2.4), (2.5)]). We found that canonical light-cone quantization misses zero modes in the presence of two-forms induced by gauge symmetry or gauge fields, while Dirac's method overcounts them, even in cases when they are not necessary, making analysis more difficult then necessary (as already pointed out by Faddeev and Jackiw [9]). We also find that both reduced phase-space quantization and Penrose quantization are the most economical method of counting zero modes, and agree with the results gotten from usual equal-time quantization.

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