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An Effective Kernel for the Bound State Problem of Positronium in Light-Cone Quantization*

Ovid C. Jacob and Alex Langnau†

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94305

We dedicate this paper to our dear friend Brian Warr.

ABSTRACT

We construct an effective kernel for the Bethe-Salpeter equation for positronium by considering up to three-particle Fock states. We also consider kernels to order α which relate the different Fock states by a system of coupled integral equations. We eliminate all but one of these equations to obtain a single equation for positronium Fock state (e^+e^-) governed by an effective kernel. We check Lorentz and gauge invariance of the theory. The integral equation is solved using hyperspherical harmonics.

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1. Introduction

Light-cone quantization is a valuable tool for understanding of the strong interaction. The main advantages of the formalism are the simpler vacuum structure [1] , the manifest boost invariance in the z-direction and the Hamiltonian formulation which leads to a very physical approach to field theory.

One of the major disadvantages of the formalism [2] (as for any Hamiltonian formulation) is its lack of manifest Lorentz invariance . This has lead to some problems with rotational invariance which will not be addressed here [3] . It has proven very succesful in many perturbative applications [4] . This paper deals with a novel solution of the basic light-cone equation

$$H|\Psi >= M^2|\Psi > \tag{1.1}$$

Here H is P^-, the Hamiltonian of the system, and Ψ is a superposition of infinite number of all the possible Fock states. There have been several proposed solutions to this equation, or equations similar to this one [5 6 7 8] . In principle, the above equation, written in full, describes a set of infinite number of coupled integral equations, the Fock states. To solve them, some sort of limiting procedure is introduced, whereby one can terminate the infinite Fock state expansion. We propose to describe here a different approach: we show that by demanding that we keep terms in the interaction piece up to some finite order of the coupling we 'naturally' cutoff this infinite expansion in the Fock states. This introduces some 'perturbative' approximation, but should have the advantage that preserves gauge invariance, since all diagrams of the same order appearing in the interaction are kept [9] . We obtain then a finite set of coupled integral equations for some of the Fock states. We than simplify this set of coupled set of integral equations and obtain one integral equation for one of the Fock state, now with and *effective* kernel. We solve this integral equation using hyperspherical harmonics, as used before [10 11] .

We then calculate the matrix elements of the electromagnetic current J^μ and of the other

Poincaré geneators. We check whether gauge and Lorentz invariance is maintained to order α, as it should be.

We then discuss how to extend this method to higher orders in the coupling.

2. O(α) truncation of the infinite Fock state expansion

For our problem, we are interested in studying e^+e^-; to first approximation, we will keep only term of order α in interaction. This means that we will only need to keep Fock states with a maximum of three particles : $|\gamma>, |\gamma\gamma>, |e^+e^->$ and $|e^+e^-\gamma>$.[12] Fock states with more particles will bring in four or more powers of e, hence will be of order α^2 or higher, which we are now ignoring[13] . We rewrite $H = H_0 + V$ where H_0 is the kinematic part of the Hamiltonian, and V is the interacting part. We obtain the following set of coupled integral equations :

$$(H_0 - M^2)|\gamma >= V_{11}|\gamma > + V_{13}|e^+e^- > + V_{14}|e^+e^-\gamma >$$

$$(H_0 - M^2)|\gamma\gamma >= V_{23}|e^+e^- >$$

$$(H_0 - M^2)|e^+e^- >= V_{31}|\gamma > + V_{32}|\gamma\gamma > + V_{33}|e^+e^- >$$

$$(H_0 - M^2)|e^+e^-\gamma >= V_{41}|\gamma >$$

The matrix elements for the interaction part as defined as follows :

$$V_{11} =< \gamma|V|\gamma >, V_{13} =< \gamma|V|e^+e^- >, V_{31} =< e^+e^-|V|\gamma >,$$
$$V_{14} =< \gamma|V|e^+e^-\gamma >, V_{41} =< e^+e^-\gamma|V|\gamma >, V_{23} =< \gamma\gamma|V|e^+e^- >,$$
$$V_{32} =< e^+e^-|V|\gamma\gamma >, V_{33} =< e^+e^-|V|e^+e^- > .$$

After easily inverting the equations for the Fock state $|e^+e^-\gamma >$, we get the following equation for $|\gamma>$:

$$|\gamma>= \frac{1}{H_0 - M^2 - V_{11} - V_{14}(H_0 - M^2)^{-1}V_{41}} V_{13}|e^+e^->$$

This can be used to obtain an equation for $|e^+e^->$:

$$(H_0 - M^2)|e^+e^->= (V_{31}\frac{1}{H_0 - M^2 - V_{11} - V_{14}(H_0 - M^2)^{-1}V_{41}} V_{13}+ \\ +V_{33})|e^+e^->$$

V_{33} contains both the one-photon exchange and the instantaneous piece. Again, we use the approximation that we keep only term of order α in the kernel and drop terms of higher powers of the coupling. Only two terms survive :

$$(H_0 - M^2)|e^+e^->= (V_{31}(H_0 - M^2)^{-1}V_{13} + V_{33})|e^+e^->$$

The first term is the annihilation piece and the second term is the elastic piece (see the definitions of the V_{ij}'s above).

There are some problems though. Let us rewrite formally :

$$|e^+e^->= \frac{1}{(H_0 - M^2)}(V_{31}(H_0 - M^2)^{-1}V_{13} + V_{33})|e^+e^->$$

Let us also write:

$$(H_0 - M^2)|e^+e^->^0 = 0$$

Then we can expand $|e^+e^->$ in terms of $|e^+e^->^0$:

$$|e^+e^->= |e^+e^->^0 + \frac{1}{(H_0 - M^2)}(V_{31}\frac{1}{(H_0 - M^2)}V_{13} + V_{33})|e^+e^->^0 +$$

$$\frac{1}{(H_0 - M^2)}(V_{31}\frac{1}{(H_0 - M^2)}V_{13} + V_{33})\frac{1}{(H_0 - M^2)}(V_{31}\frac{1}{(H_0 - M^2)}V_{13} + V_{33})|e^+e^->^0$$

Now if we count powers of α we find that V_{13} and V_{31} go as $\alpha^{1/2}$ while V_{33} goes as α, so the last term is of order α^2. This is a troublesome term. Note that iteration always occurs such that there are never two photons in flight - there can't be since in our original kernel there was only one photon in flight. BUT, we know that there are terms in V, to order α^2 which HAVE two photons on flight. Thus, by iterating our kernel we don't get the full interaction. In this sense we don't really have gauge invariance. A possible way to deal with this problem is to compute an effective four fermion interaction which is just this O(α^2) term with two photons in flight by integrating out the internal momentum. We then can add this to our original interaction [14] .

. The instantaneous piece for the fermions is a bit more problematic and we are not quite sure what to do. The reason this piece is important is that when we iterate the equation, like we did above, we need to have these instantaneous pieces which are naturally generated by higher order terms coming from V.

There might also be a logarithmic singularity coming from the the Fock state truncation, but wew were not able to see it in our numerical work so far.

3. Solving for the e^+e^- Fock state

We work now in the rest frame of the center of mass of the e^+e^- pair, hence the fermions have momenta $p_{e^+} = (x_1P, p^-_{e^+}, k_\perp)$ and $p_{e^-} = (x_2P, p^-_{e^-}, -k_\perp)$, where the x_i's are the fraction of total momentum P carried by each individual fermion, and $k_\perp$ is the relative momemtum. We define the positronium amplitude as follows :

$$\Psi_{\lambda\mu}(k_\perp, x_i) = \frac{\overline{v}_\lambda(-k_\perp, x_2)}{\sqrt{x_2}} \frac{u_\mu(k_\perp, x_1)}{\sqrt{x_1}} \phi(k_\perp, x_i)$$

After doing the spinor algebra [15] we get the following form for the positronium amplitude ϕ

$$(M^2 - \frac{k_\perp^2 + m^2}{x_1 x_2})\phi(k_\perp, x_i) = 4\pi\alpha \int \frac{d^2 l_\perp [dy]}{16\pi^3} (\frac{-m^2}{M^2}) V_{1331} \phi(l_\perp, y_i) +$$

$$4\pi\alpha \int \frac{d^2 l_\perp [dy]}{16\pi^3} V_{331} \left\{ \frac{\theta(x_1 - y_1)}{x_1 - y_1} \frac{m^2}{M^2 - \frac{k_\perp^2 + m^2}{x_1} - \frac{l_\perp^2 + m^2}{y_2} - \frac{(k_\perp - l_\perp)^2}{x_1 - y_1}} + (1 \leftrightarrow 2) \right\} \phi(l_\perp, y_i),$$

where V_{1331} and V_{331} are factors from the spinor algebra:

$$V_{1331} = \frac{1}{x_1 x_2}(1 - \frac{k_\perp^2}{m^2}) + \frac{1}{y_1 y_2}(1 - \frac{l_\perp^2}{m^2}) + \frac{1}{x_1 y_1}(1 + \frac{k_\perp . l_\perp}{m^2}) + \frac{1}{x_2 y_2}(1 + \frac{k_\perp . l_\perp}{m^2})$$

and

$$V_{331} = 16\frac{y^2 \frac{k_\perp^2}{m^2} - y^2 - 2xy\frac{k_\perp . l_\perp}{m^2} - 2xy + x^2 \frac{l_\perp^2}{m^2} - x^2 - \frac{k_\perp^2}{m^2} - 6\frac{k_\perp . l_\perp}{m^2} - \frac{l_\perp^2}{m^2} - 4}{(1-y^2)(1-x^2)}$$

The notation y is for $y_1 - y_2$ and the abbreviation $[dy]$ is

$$[dy] = \frac{dy_1 dy_2 \delta(1 - y_1 - y_2)}{y_1 y_2}$$

.

We map the phase space of $k_\perp$ and $x = x_1 - x_2$ to a three-dimensional space by introducing a third component for the two-vector $k_\perp$:

$$k_3 = -x\sqrt{\frac{k_\perp^2 + m^2}{1 - x^2}}.$$

The integral equation looks like this :

$$\left(\vec{q}^2 + m^2 - \frac{M^2}{4}\right)\phi(\vec{k}) = \frac{\alpha}{4\pi^2}\int \frac{d^3 l}{E(l)} K(\vec{k}, \vec{l}, M^2) V(\vec{k}, \vec{l})\phi(\vec{l})$$

Here E is the energy $E(l) = \sqrt{l^2 + m^2}$, K is the effective kernel, V is the spinor factor in these new variables. V is some rational function of these new variables, but K the kernel is of the following form :

$$K(\vec{k}, \vec{l}) = \left[(\vec{k} - \vec{l})^2 + R(\vec{k}, \vec{l}, M^2)\right]^{-1}$$

and $R(\vec{k}, \vec{l}, M^2)$ is like Sawicki[10]. As he noted, this is very similar to the non-relativistic Lippman-Schwinger equation[10,11]. The main difference is the presence of the non-local term R. Nontheless, we might be able to apply some ideas from solving the L-S equation. Fock[16] was the first to map three-dimensional integral equation to a four-dimensional one and then use the orthonormal functions in that space , the hyperspherical harmonics to solve that problem. To do that he mapped the three-dimensional space R^3 onto the four-sphere S^3. There are several reasons for doing so. First, it seems that the Coulomb singularity can be handled rather easily in this way. Second, this allows us to use orthormal functions which have elegant analytic properties. Define the following :

$$(k_0/m)^2 = D_2/m, D_2 = D - D^2/4m, D = 2m - M.$$

$$k_\perp = k(\cos\phi, \sin\phi)$$

Then the orthonormal vector on S^3 is :

$$\xi_0 = \frac{k_0^2 - k^2}{k_0^2 + k^2}, \vec{\xi} = \frac{k_0 \vec{k}}{k_0^2 + k^2} = \sin\alpha(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The phase space in these new coordinates is :

$$d\Omega = \sin\alpha^2 d\alpha \sin\theta d\theta d\phi$$

$$d\Omega = \frac{2k_0}{(k_0^2 + k^2)^3} d^3k$$

Following Fock and Sawicki, we can relate the three-vectors to the four-sphere vectors as well :

$$(\vec{k} - \vec{k}_1)^2 = \frac{(k_0^2 + k^2)(k_0^2 + k_1^2)}{4k_0^2} (|\xi - \xi_1|)^2$$

For the Lippman-Schwinger equation, define the HSH amplitude

$$\Psi(\Omega) = \frac{1}{4k_0^{5/2}} (k_0^2 + k^2)^2 \phi(\vec{k})$$

to obtain the following equation for Ψ :

$$\Psi(\Omega) = \frac{\lambda}{4\pi} (\frac{m}{k_0}) \int d\Omega_1 \frac{1}{|\xi - \xi_1|^2} \Psi(\Omega_1)$$

Here λ is $\frac{\alpha}{\pi}$. As Sawicki points out in[10], the Lippman-Schwinger equaition is diagonal in the HSH basis. Clearly this simpifies the calculations greatly.

Let us define now HSH amplitude $\Psi(\Omega)$ for our positronium problem :

$$\Psi(\Omega) = C(k_0^2 + k^2)^2(1-x^2)^{-1/4}\phi(k_\perp, x_i)$$

In terms of this HSH amplitude, the integral equation for the e^+e^- Fock state is :

$$\Psi(\Omega) = \frac{\alpha}{32\pi^2}(\frac{k_0}{m})(-\frac{m^2}{M^2})\int d\Omega_1 \frac{1+\tan^2\alpha/2}{(1+\frac{k_0^2}{m^2}\tan^2\alpha/2)^{1/4}} \frac{1+\tan^2\alpha_1/2}{(1+\frac{k_0^2}{m^2}\tan^2\alpha_1/2)^{1/4}} K_{1331}\Psi(\Omega_1)$$

$$+\frac{\alpha}{32\pi^2}4(m/k_0)\int d\Omega_1 \frac{1}{(1+\frac{k_0^2}{m^2}\tan^2\alpha/2)^{1/4}} \frac{1}{(1+\frac{k_0^2}{m^2}\tan^2\alpha_1/2)^{1/4}} K_{331}\Psi(\Omega_1),$$

where the K's are the spinor factors in HSH notation.

4. Calculating with Hyperspherical Harmonics

To solve the integral equation we follow Sawicki[10]. We expand first the positronium amplitude in HSH :

$$\Psi(\Omega) = \sum_{nlm} a_{nlm} Y_{nlm}(\Omega)$$

where the Y_{nlm}'s are HSH. We use the orthonormal properties of the HSH as well as the reexpression of products of HSH in terms of 9J coefficients and HSH and transform the integral equation for Ψ into an eigenvalue problem for the a_{nlm}'s:

$$a_{nlm} = \frac{\lambda}{16\pi}(\frac{m}{k_0}) \sum_{n_1 l_1 m} d_{nlm;n_1 l_1 m}(k_0/m) a_{n_1 l_1 m}.$$

The heart of the calculation is in obtaining fast, stable results for the d's. They are define to be :

$$d_{nlm;n_1l_1m_1} = \sum_{n_2l_2m_2} \sum_{n_3l_3m_3} \sum_{n_4l_4m_4}$$

$$\sqrt{(2l_2+1)(2l_3+1)n_2n_3n/2\pi^2}\langle l_2m_2, l_3m_3||lm\rangle$$

$$\sqrt{(2l_2+1)(2l_4+1)n_2n_4n/2\pi^2}\langle l_2m_2, l_4m_4||l_1m_1\rangle$$

$$C_{9J}(\frac{1}{2}(n_2-1),\frac{1}{2}(n_2-1),l_2,\frac{1}{2}(n_3-1),\frac{1}{2}(n_3-1),l_3,\frac{1}{2}(n-1),\frac{1}{2}(n-1),l)$$

$$C_{9J}(\frac{1}{2}(n_2-1),\frac{1}{2}(n_2-1),l_2,\frac{1}{2}(n_4-1),\frac{1}{2}(n_4-1),l_4,\frac{1}{2}(n_1-1),\frac{1}{2}(n_1-1),l_1)$$

$$C_{n_3 l_3 m_3, n_4 l_4 m_4}(k_0/m)$$

The factor λ is α/π like in the Lippman-Schwinger case. The C_{9J}'s are the 9J coefficients and the factors enclosed in the $\langle\rangle$ are the 3J coefficients. The function $C_{nlm,n'l'm'}$ are defined like in Sawicki :

$$C_{nlm,n'l'm'}(k_0/m)=\int d\Omega\int d\Omega_1 Y^*_{nlm}(\Omega)r(\xi,\xi_1,k_0/m)Y_{n'l'm'}(\Omega_1)$$

and r is

$$r(\xi,\xi_1,k_0/m)=r_{annihil}(\xi,\xi_1,k_0/m)+r_{elast}(\xi,\xi_1,k_0/m)$$

and the two pieceses are defined thus :

$$r_{annihil}(\xi,\xi_1,k_0/m)=(\frac{k_0}{m})\left(\frac{-m^2}{M^2}\right)g(\xi,k_0/m)|\xi-\xi_1|^2 g(\xi_1,k_0/m)K_{1331}$$

where $g(\xi,k_0/m)$ is

$$g(\xi,k_0/m)=\frac{(1+\tan^2\alpha/2)^2}{(1+\frac{k_0^2}{m^2}\tan^2\alpha/2)^{1/4}}$$

and the 'elastic' piece - the piece corresponding to the contribution from V_{33}, the interaction connecting e^+e^- to itself :

$$r_{elast}(\xi,\xi_1,k_0/m)=4(\frac{m}{k_0})f(k)\frac{|\xi-\xi_1|^2}{|\xi-\xi_1|^2+\Delta(\xi,\xi_1,k_0/m)}f(k_1)K_{331}$$

$|\xi-\xi_1|^2$ and $\Delta(\xi,\xi_1,k_0/m)$ are :

$$|\xi - \xi_1|^2 = 2\big(1 - \cos\alpha\cos\alpha_1 - \sin\alpha\sin\alpha_1\cos\theta\cos\theta_1 - \sin\alpha\sin\alpha_1\sin\theta\sin\theta_1\cos(\phi-\phi_1)\big)$$

and

$$\begin{aligned}\Delta(\xi,\xi_1,k_0/m) = -\left\{\frac{f(k)}{f(k_1)} + \frac{f(k_1)}{f(k)} - 2\right\}\sin\alpha\sin\alpha_1\cos\theta\cos\theta_1 + \\ 2\frac{k_0}{m}\left\{\frac{1}{1+\tan^2\alpha/2} + \frac{1}{1+\tan^2\alpha_1/2}\right\}\times \\ |f(k)(1+\tan^2\alpha/2)\sin\alpha\cos\theta - f(k_1)(1+\tan^2\alpha_1/2)\sin\alpha_1\cos\theta_1|\end{aligned}$$

where

$$f(k) = [1 + (k_0/m)^2\tan^2\alpha/2]^{-1/2}$$

5. Lorentz and Gauge Invariance

We can check now whether continuous symmetries like rotational and gauge invariance are preserved. We can calculate the matrix elements of the electromagnetic current J^μ and of the other Poincaré generators. We can check whether gauge and Lorentz invariance is maintained to order α, as it should be.

Let us consider J^μ in the Fock space spanned by truncating operators to order α. We want to compute the matrix elements

$$q_\mu < \alpha(p)|J^\mu(q)|\beta(l)>, \quad q = p - l$$

and verify that it is zero for the Fock states $\alpha, \beta = |\gamma>, |\gamma\gamma>, |e^+e^->, |e^+e^-\gamma>$. Then, to O($\alpha$), the matrix elements of the electromagnetic current :

$$q_\mu < \gamma(p)|J^\mu(q)|e^+e^-(l)>, q_\mu < e^+e^-(p)|J^\mu(q)|e^+e^-(l)>$$

and their Hermitian conjugates should be zero.

Let us calculate this for l being the rest frame of positronium, the photon with momentum $q_\mu = (0,0,q_\perp)$ such that $q^2 = -Q^2$. Then the final momentum of positronium is $p = (1, Q^2+M^2, q_\perp)$. We get the following result for $\mu = +$

$$< e^+e^-(p)|J^+(q)|e^+e^-(l) >= -\sqrt{4\pi\alpha}(1 - \frac{iq_2}{m} < x_1x_2 >) =$$

$$-e\int \frac{d^2k_\perp}{16\pi^3}[dx]\phi^*(k_\perp,x_i)\phi(k_\perp,x_i)\{\frac{16}{x_1x_2}\frac{16iq_2}{m}\}$$

For $\mu = -$ we get the folowing form :

$$< e^+e^-(p)|J^-(q)|e^+e^-(l) >= -\sqrt{4\pi\alpha}(< \frac{1}{x_1x_2} > -\frac{iq_2}{m} - \frac{q_2^2}{m^2} < x_1x_2 >) =$$

$$-e\int \frac{d^2k_\perp}{16\pi^3}[dx]\phi^*(k_\perp,x_i)\phi(k_\perp,x_i)\{\frac{16(k_\perp^2+m^2)}{m^2x_1x_2} - \frac{16iq_2}{mx_1x_2} - \frac{16iq_2^2}{m^2} - \}$$

where we have ignored terms like $< \frac{k^2}{m^2} >$ which are of order α^2 .

Finally, for $\mu = i$ we get

$$< e^+e^-(p)|J^i(q)|e^+e^-(l) >= -\sqrt{\pi\alpha}\frac{q_i}{m} =$$

$$-e\frac{q^i}{m}\int \frac{d^2k_\perp}{16\pi^3}[dx]\phi^*(k_\perp,x_i)\phi(k_\perp,x_i)\{\frac{8}{x_1x_2}\}$$

Now, if we demand rotational invariance at the marix element level, then we should have

$$[< J^1 >, < J^2 >] = i < J^3 >$$

which, since the i-th components of the matrix elements of the electromagnetic current differ just by having different components of q, means that they commute, so the third component must have zero matrix elemnt. Then, this means that

$$< J^+ > = < J^- >$$

which for our case means that we get the following two integrals for rotational invariance to hold

$$1 =< \frac{1}{x_1 x_2} > - \frac{q_2^2}{m^2} < \frac{1}{x_1 x_2} >$$

plus

$$< x_1 x_2 >= 1$$

Clearly neither of these equations can be satisfied, so this approach breaks rotational invariance at the matrix element level. We'll see in the next section how badly.

Likewise, we check gauge invariance by computing the matrix element of the electromagnetic current . We get the following

$$q_\mu < e^+ e^-(p)|J^\mu(q)|e^+ e^-(l) >= -\sqrt{\pi\alpha}\frac{Q^2}{m^2} =$$

$$\frac{-eQ^2}{m^2} \int \frac{d^2 k_\perp}{16\pi^3}[dx]\phi^*(k_\perp, x_i)\phi(k_\perp, x_i)\{\frac{8}{x_1 x_2}\}$$

This is also not zero, so gauge invariance is also broken at the matrix element level.

6. Results

First we calculated the ground state. As is known[17] , the positronium ground state has a singlet (parapositronium) and a triplet (orthopositronium) value. To leading order , they are degenarate :

$$M^2 = 4m^2 \left\{ \frac{\alpha^2}{4n^2} \right\}$$

The calculation proceedes as follows : We pick a value for α and thus implicitly for the 'theoretical' eigenvalue λ; we calculate $d_{nlm;n'l'm'}$ up to some value of the hyperspherical number n .

Now things get a bit tricky. Note that since $\alpha = \lambda\pi$, the mass M depends on the eigenvalue . Furthermore, to make things explicit, the matrix $d_{nlm,n'l'm'}$ depends on M

, hence on λ via its dependence on k_0/m. If we put in the λ dependence explictly, the eigenvalue problem for the coefficients a becomes :

$$a_{nlm} = \frac{\lambda}{16\pi} \sum_{n_1 l_1 m} \big(1 - \frac{M^2(\lambda)}{4m^2}\big)^{-1/2} d_{nlm;n_1 l_1 m}(M(\lambda), m) a_{n_1 l_1 m}.$$

since

$$\big(\frac{k_0}{m}\big) = \big(1 - \frac{M^2}{4m^2}\big)^{1/2}$$

We solve now for the eigenvalue λ_{calc} for a fixed value λ_{th} in the modified equation

$$a_{nlm} = \frac{\lambda_{calc}}{16\pi} \sum_{n_1 l_1 m} \big(1 - \frac{M^2(\lambda_{th})}{4m^2}\big)^{-1/2} d_{nlm;n_1 l_1 m}(M(\lambda_{th}), m) a_{n_1 l_1 m}.$$

We calculate the a's and λ_{calc} for various values of n and notice that as we increase n we get closer to λ_{th}. The calculation converges for n=8. Imposing parity symmetry, we can pick states of either l even or odd, thereby simplifying the calculation considerably - we need fewer matrix elements. Then there are only 454 matrix elements to compute. In the first pass, we look only at small values of α. We used the VEGAS algorithm[18] with only 10,000 points and got within 10 percent of the exact of the 'theoretical' result. For $\alpha = 0.3, 0.5, 0.7$, $\lambda_{th} = 0.09549, 0.1591, 0.2864$ while $\lambda_{calc} = 0.09236, 0.1248, 0.2394$. With our solutions we get $< \frac{1}{x_1 x_2} >= 4.1686$ and $< x_1 x_2 >= .2417$. (As a parenthetical remark, let us note that $x_1 x_2 = 1/4(1-x^2)$, with $x = x_1 - x2$, so we get that $< x^2 >= 0$.) Rotational invariance is supposed to be 1.0 and 0.2417 theoretically, compared with the calculated values of 4.1686- $\frac{q_2^2}{m^2}$0.2417 and 1.0 (the two integrals of the last section for rotational invariance). Both conditions are violated very badly . For gauge invariance, should have 0.0 theoretically compared with $\sqrt{\pi\alpha}(Q^2/m^2)$ the calculated value. We don't know how well other calculations obey rotational or gauge invariance, hence can't make a judgement on the 'goodness' or 'badness' of this calculation, though the conditions for rotational invariance does seem to be quite off.

We should add here that the lack of gauge invariance means that there is an extra $1/p^+$ singularity coming from the $1/\eta.k$ piece in the photon propagator. We were not able to see this in our calculations.

By way of comparison, M. Krautgärtner,Pauli and Wölz[7] need at least 225x225 matrices - that's 48,625 matrix elements- to compute effects admitedly to higher order (α^2). The difference is that in the DLCQ approach the evaluation of the matrix elements is rather trivial; still, the matrix gets big quite soon. Another similar calculation in DLCQ, is that of Kaluza and Pauli[6], which is a different approximation for the full Hamiltonian. They also need rather large matrices - up to 81,668 matrix elements. We want to emphasize that our method produces rather good results with a small number of matrix elements.

The calculation presented here is very encouraging and shows that we can go in more detail. The first step will be to compute the singlet and the triplet masses M_s and M_t to the next order where we'll see the hyperfine split. We can do this even if we keep the operators only up to order α, but we need to impliment charge symmetry. One of us (O.C. J.) will address this issue in another paper[19] .

7. Conclusions and Comments

It seems that by truncating the operators to some order in the coupling we obtain a great simplification in the solution of the eigenvalue problem for the spectrum of the theory. There are still some issues to be clarified regarding rotational and gauge invariance. One of us (O.C.J.) will address this in a future paper in a different approach in which discrete symmetries are implimented[19]. Yet it is a promising approach. Furthermore, this method is easily generalizable.

An obvious next step is to go then to O(α^2) in the operators. This can be done easily. First of all, we need now to go to five-particle Fock states: $|\gamma>$, $|\gamma\gamma>$, $|e^+e^->$, $|\gamma\gamma\gamma>$ $|e^+e^-\gamma>$, $|\gamma\gamma\gamma\gamma>$, $|e^+e^-e^+e^->$, $|e^+e^-\gamma\gamma>$, $|e^+e^-e^+e^-\gamma>$. As before, we compute now the various matrix elements with operators to order α^2 .

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