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# Discrete Symmetries for the Bound State Problem of Positronium in Light-Cone Quantization*

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## ABSTRACT

I use as the starting point the effective kernel for positronium which Alex Langnau and I constructed to impliment discrete symmetries. It is found that implimenting charge conjugation makes obvious rotational and gauge symmetry violation in the Hamiltonian which can be fixed by adding counterterms to the Hamiltonian. This induces charge symmetry violation as well as rotational symmetry violation in the electromagnetic current matrix elements. Gauge symmetry is found to be maintained .

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# 1. Introduction

In this paper I continue the study of positronium which Alex Langnau and I started,[1] which presented a novel solution to the equation

$$H|\Psi>= M^2|\Psi> \tag{1.1)1}$$

Here H is $P^-$, the Hamiltonian of the system, and $\Psi$ is a superposition of infinite number of all the possible Fock states.

Discrete symmetry of charge conjugation is implimented so as to compute the single-triplet mass splitting. It is found that there are rotational and gauge invariance violating pieces in the interacting part of the Hamiltonian which can be fixed by adding appropiate counterterms to the Hamiltonian . Then the effect this has on Lorentz conservation is calculated by computing the rotational algebra of the electromagnetic current, as well as computing the divergence of this current, which should vanish by gauge invariance.

# 2. O($\alpha$) truncation using charge conjugation

It is observed that charge symmetry , a symmetry of the Hamiltonian, simplifies things a bit : states which go via an even number of photon are even under charge conjugation and connect to the charge conjugation even piece of positronium ( parapositronium); those going via an odd number of photon are odd under charge conjugation and connect to orthopositronium.

Note now that there are two sets of Fock states, charge conjugation (C) even states : $|\gamma\gamma>, |e^+e^- >_{para}$ ; and C-odd states: $|\gamma>, |e^+e^- >_{ortho}$ and $|e^+e^-\gamma>$.

This can be used to obtain an equation for $|e^+e^- >_{para}$ :

$$(H_0 - M^2)|e^+e^- >_{para}= (V_{32}\frac{1}{H_0 - M^2 - V_{22} - V_{24}(H_0 - M^2 - V_{44})^{-1}V_{42}}V_{23}+$$

$$V_{34}\frac{1}{H_0 - M^2 - V_{44}}V_{43} + +V_{33})|e^+e^- >_{para}$$

Again, using the approximation that only term of order $\alpha$ are kept in the kernel and droping terms of higher powers of the coupling, only three terms survive :

$$(H_0 - M^2)|e^+e^- >_{para}= (V_{32}(H_0 - M^2)^{-1}V_{23} + V_{34}(H_0 - M^2)^{-1}V_{43} + V_{33})|e^+e^- >_{para}$$

The first term gives instantaneous fermion excheange, the second term the self-energy pieces for the fermions plus the one-photon exchange piece; the last term is the instantaneous one-photon exhange. The instantaneous fermion pieces arising from the first piece is not included since it is of order O($\alpha^2$). (see the definitions of the $V_{ij}$'s in[1]).

A slightly different equation for orthopositronium is obtained due to the presence of the annihilation piece, which was absent for the parapositronium piece, due to the charge conjugation symmetry :

$$(H_0 - M^2)|e^+e^- >_{ortho}= (V_{31}\frac{1}{H_0 - M^2 - V_{11} - V_{14}(H_0 - M^2 - V_{44})^{-1}V_{41}}V_{13}+$$

$$V_{34}\frac{1}{H_0 - M^2 - V_{44}}V_{43} + +V_{33})|e^+e^- >_{ortho}$$

This becomes, to leading order in $\alpha$

$$(H_0 - M^2)|e^+e^- >_{ortho}= (V_{31}(H_0 - M^2)^{-1}V_{13} + V_{34}(H_0 - M^2)^{-1}V_{43} + V_{33})|e^+e^- >_{ortho}$$

Here the first term is the annihilation piece, the second term gives the fermion self-energy contributions and the last term gives the photon instantaneous piece.

## 3. Solving for the $e^+e^-$ charge conjugation Fock state

As in[1], in the rest frame of the center of mass of the $e^+e^-$ pair the fermions have momenta $p_{e^+} = (x_1P, p^-_{e^+}, k_\perp)$ and $p_{e^-} = (x_2P, p^-_{e^-}, -k_\perp)$, where the $x_i$'s are the fraction of total momentum P carried by each individual fermion, and $k_\perp$ is the relative momemtum.

Define the positronium amplitude as follows :

$$\Psi(k_\perp, x_i) = \frac{\overline{v}_\lambda(-k_\perp, x_2)}{\sqrt{x_2}} \frac{u_\mu(k_\perp, x_1)}{\sqrt{x_1}} \phi_{\lambda\mu}(k_\perp, x_i)$$

where $\lambda, \mu = \uparrow, \downarrow$. Implimenting C, the following forms are obtained for parapositronium :

$$\Psi(k_\perp, x_i)_{para} = \left\{ \overline{v}_\uparrow u_\downarrow - \overline{v}_\downarrow u_\uparrow \right\} \frac{\phi_{para}(k_\perp, x_i)}{\sqrt{2x_1x_2}}$$

Orthopositronium is somewhat complicated :

$$\Psi(k_\perp, x_i)_{ortho} = \left\{ \overline{v}_\uparrow u_\downarrow + \overline{v}_\downarrow u_\uparrow \right\} \frac{\phi^0_{ortho}(k_\perp, x_i)}{\sqrt{2x_1x_2}}$$

$$+\left\{ \overline{v}_\uparrow u_\uparrow \right\} \frac{\phi^{+1}_{ortho}(k_\perp, x_i)}{\sqrt{x_1x_2}}$$

$$+\left\{ \overline{v}_\downarrow u_\downarrow \right\} \frac{\phi^{-1}_{ortho}(k_\perp, x_i)}{\sqrt{x_1x_2}}$$

A simple equation is obtained for parapositronium, a set of coupled integral equations for the components of orthopositronium. Let us study parapositronium first.

After doing the spinor algebra , the following form is obtained for the parapositronium amplitude $\phi_{para}$

$$(M^2 - \frac{k_\perp^2 + m^2}{x_1x_2} - \delta M_s^2(k_\perp, x_i)) \frac{\phi_{para}(k_\perp, x_i)}{x_1x_2} =$$

$$\frac{4\pi\alpha}{4} \int \frac{dl_\perp[dy]}{16\pi^3} V_{331} \left\{ \frac{\theta(x_1 - y_1)}{x_1 - y_1} \frac{m^2}{M^2 - \frac{k_\perp^2 + m^2}{x_1} - \frac{l_\perp^2 + m^2}{y_2} - \frac{(k_\perp - l_\perp)^2}{x_1 - y_1}} + (1 \leftrightarrow 2) \right\} \phi_{para}(l_\perp, y_i),$$

where $\delta M_s^2$ is the term coming from the (anti)-fermion self-energy and $V_{331}$ is a factor from the spinor algebra:

$$V_{331} = V_{331}^{odd} + V_{331}^{even}$$

where

$$V_{331}^{odd} = 16\frac{y^2\frac{k_\perp^2}{m^2} - x^2\frac{l_\perp^2}{m^2} + y^2 - x^2 - \frac{k_\perp^2}{m^2} + \frac{l_\perp^2}{m^2}}{(1-y^2)(1-x^2)(y-x)} = A$$

and

$$V_{331}^{even} = 8\frac{y^2\frac{k_\perp^2}{m^2} + x^2\frac{l_\perp^2}{m^2} - y^2 - x^2 - \frac{k_\perp^2}{m^2} - \frac{l_\perp^2}{m^2} - 2xy\frac{k_\perp . l_\perp}{m^2} - 2}{(1-y^2)(1-x^2)}$$

The odd and even superscripts refer to symmetry under parity , which for this case means symmetry under the interchange $k_\perp \leftrightarrow -k_\perp$ and $x \leftrightarrow -x$ and similarly for $l_\perp$ and $y$.

Notice that there two types of singularities : one induced by breakdown of rotational invariance, in $\delta M_s^2$, as Langnau and Burkhardt pointed out [2] ; and one due to breakdown of gauge invariance, coming from the fact that not all the external legs are on energy shell - the fermions in the integral equation are on mass shell but **not** on energy shell. It is tempting - and probably wrong - to equate these two singularities due to their similar $1/p^+$ structure. What I'll suggest instead is to add appropiate counterterms in the Hamiltonian to cancel these pieces :

$$H -> H + \delta M_s^2(k_\perp, x_i) - K_{331}(k_\perp, x_i)$$

where the rotational counterterm with a photon mass $\lambda$ is [2]

$$\delta M_s^2(k_\perp, x_i)) = \int \frac{d^2 l_\perp [dy]}{16\pi^3}\left\{\frac{yk^2 + m^2 - \lambda^2 + l_\perp^2(1-y)}{-l_\perp^2 - m^2 y + y(1-y)k^2 - \lambda^2(1-y)}\right\}$$

For the gauge counterterm $K_{331}$ the result is

$$K_{331}(k_\perp, x_i) = \int \frac{d^2 l_\perp [dy]}{16\pi^3} V_{331}^{odd}\left\{\frac{\theta(x_1 - y_1)}{x_1 - y_1}\frac{m^2}{M^2 - \frac{k_\perp^2 + m^2}{x_1} - \frac{l_\perp^2 + m^2}{y_2} - \frac{(k_\perp - l_\perp)^2}{x_1 - y_1}} - (1 \leftrightarrow 2)\right\}$$

Note the minus sign in the kernel K.

For orthopositronium, the following set of coupled integral equations are obtained :

$$\{(M^2-\frac{k_\perp^2+m^2}{x_1x_2})\delta^{\alpha\beta}-\delta M_t^2(k_\perp,x_i)^{\alpha\beta}\}\frac{\phi^\beta_{ortho}(k_\perp,x_i)}{x_1x_2}=\frac{-4\pi\alpha}{4}\int\frac{d^2l_\perp[dy]}{16\pi^3}(\frac{-m^2}{M^2})V^{\alpha\beta}_{3113}\phi^\beta_{ortho}(l_\perp,y_i)+$$

$$\frac{-4\pi\alpha}{4}\int\frac{d^2l_\perp[dy]}{16\pi^3}V^{\alpha\beta}_{331}\left\{\frac{\theta(x_1-y_1)}{x_1-y_1}\frac{m^2}{M^2-\frac{k_\perp^2+m^2}{x_1}-\frac{l_\perp^2+m^2}{y_2}-\frac{(k_\perp-l_\perp)^2}{x_1-y_1}}+(1\leftrightarrow 2)\right\}\phi^\beta_{ortho}(l_\perp,y_i),$$

and here the $\alpha,\beta$ indices stand for the various forms -1, 0,+1 which form the orthopositronium state. The kernels $V^{\alpha\beta}_{3113}$ and $V^{\alpha\beta}_{331}$ are more numerous and complicated. Here are some typical forms :

$$V_{3113}^{+0,odd}=4\sqrt{2}\frac{(y^2\frac{k_1}{m}-x^2\frac{l_1}{m}+yx\frac{k_1}{m}+yx\frac{l_1}{m}-2\frac{k_1}{m}+2\frac{l_1}{m})}{(1-y^2)(1-x^2)}=B$$

and

$$V_{3113}^{+0,even}=4\sqrt{2}\frac{(y\frac{k_1}{m}+y\frac{l_1}{m}+x\frac{k_1}{m}+x\frac{l_1}{m})}{(1-y^2)(1-x^2)}$$

or

$$V_{33}^{00,odd}=16\frac{(y^2\frac{k_\perp^2}{m^2}-x^2\frac{l_\perp^2}{m^2}+y^2-x^2-\frac{k_\perp^2}{m^2}+\frac{l_\perp^2}{m^2})}{(1-y^2)(1-x^2)(y-x)}=A$$

and

$$V_{331}^{00,even}=8\frac{(y^2\frac{k_\perp^2}{m^2}+y^2-2yx\frac{k_\perp.l_\perp}{m^2}-4yx+x^2\frac{l_\perp^2}{m^2}+x^2-\frac{k_\perp^2}{m^2}-\frac{l_\perp^2}{m^2}-2\frac{k_\perp.l_\perp}{m^2}-2)}{(1-y^2)(1-x^2)}$$

As before, the odd, even superscripts refer to symmetry under parity of the respective terms. The parity odd (really gauge symmetry breaking) piece is a matrix with zero entries everywhere except

$$V_{3113}^{+0,odd}=V_{3113}^{0-,odd}=B$$

$$V_{3113}^{0+,odd}=V_{3113}^{-0,odd}=-B$$

$$V_{331}^{00,odd}=V_{331}^{++,odd}=V_{331}^{--,odd}=-A$$

As in the para case, a distinction is made between infinities due to breakdown of rotational invariance, showing up in $\delta M_t^2$ from those coming from breakdown of gauge invariance.

Further counterterms are added to render all the integral equations finite. It is 'satisfying', if one can say this about infinities, that the same counter terms $\delta M^2$ and A account for some of the singularities for both para and ortho states. There are the singular pieces denoted by B which are new for ortho, due to the possibility of coupling between components of the ortho state. Schematically, the Hamiltonian is modified by the rotational counterterm $\delta M^2$ and the rotational counterterms $K_{331}$ , $L_{331}$ and $L_{311}$

$$H->H+\delta M^2(k_\perp,x_i)-K_{331}(k_\perp,x_i)-L_{331}(k_\perp,x_i)-L_{311}(k_\perp,x_i)$$

$$L_{311}(k_\perp,x_i)=\frac{-4\pi\alpha}{4}\int\frac{d^2l_\perp[dy]}{16\pi^3}(\frac{-m^2}{M^2})V_{3113}^{\alpha\beta,odd}$$

$$L_{331}(k_\perp,x_i)=\frac{-4\pi\alpha}{4}\int\frac{d^2l_\perp[dy]}{16\pi^3}V_{331}^{\alpha\beta,odd}\left\{\frac{\theta(x_1-y_1)}{x_1-y_1}\frac{m^2}{M^2-\frac{k_\perp^2+m^2}{x_1}-\frac{l_\perp^2+m^2}{y_2}-\frac{(k_\perp-l_\perp)^2}{x_1-y_1}}-(1\leftrightarrow 2)\right\}$$

## 4. Symmetries, Discrete and Continuous

One can check whether discrete symmetries like charge conjugation is preserved; similarly for continuous symmetries like rotational and gauge invariance by calculating the matrix elements of the electromagnetic current $J^\mu$ . One can also check whether gauge and Lorentz invariance is maintained to order $\alpha$, as it should be.

Let me consider $J^\mu$ in the Fock space spanned by truncating operators to order $\alpha$. It is desired to compute the matrix elements

$$q_\mu<\alpha(p)|J^\mu(q)|\beta(l)>,\quad q=p-l$$

and verify that it is zero for the Fock states $\alpha,\beta=|\gamma>,|\gamma\gamma>,|e^+e^->,|e^+e^-\gamma>$. I can use charge conjugation to simplify this problem a bit. Then, to O($\alpha$), the matrix elements of the electromagnetic current :

$$q_\mu<\gamma(p)|J^\mu(q)|e^+e^-(l)>,q_\mu<e^+e^-(p)|J^\mu(q)|e^+e^-(l)>$$

and their Hermitian conjugates should be zero.

Now, the electromagnetic current $J^\mu$ is odd under charge conjugation[3] , so it will have non-trivial matrix elements only between states of opposite charge conjugation ; it should give zero for matrix elements between states of same charge conjugation. Thus, the following matrix elements

$$< e^+e^-(p)|_{para} J^\mu(q)|e^+e^-(l) >_{para}$$

$$< e^+e^-(p)|_{ortho} J^\mu(q)|e^+e^-(l) >_{ortho}$$

vanish by charge conjugation for all values of $\mu$. For rotational invariance, the + and - components of the current $J^\mu$ should give the same results[1]:

$$< e^+e^-(p)|_{para} J^+(q)|e^+e^-(l) >_{para} =$$

$$< e^+e^-(p)|_{para} J^-(q)|e^+e^-(l) >_{para}$$

and

$$< e^+e^-(p)|_{ortho} J^+(q)|e^+e^-(l) >_{ortho} =$$

$$< e^+e^-(p)|_{ortho} J^-(q)|e^+e^-(l) >_{ortho}$$

On the other hand, the matrix element

$$q_\mu < e^+e^-(p)|_{ortho} J^\mu(q)|e^+e^-(l) >_{para}$$

should vanish due to gauge invariance.

Let us check charge symmetry first. Let l be in the rest frame of positronium, the photon with momentum $q_\mu = (0, 0, q_\perp)$ such that $q^2 = -Q^2$. Then the final momentum of positronium is $p = (1, Q^2 + M^2, q_\perp)$. The following result for $\mu = +$ follow, upon normalization

$$< e^+e^-(p)|_{para} J^+(q)|e^+e^-(l) >_{para} = -2\sqrt{4\pi\alpha}$$

$$-e \int \frac{d^2k_\perp}{16\pi^3}[dx]\{\frac{8}{x_1x_2}\}\phi^*_{para}(k_\perp, x_i)\phi_{para}(k_\perp, x_i)$$

Clearly, this is also not zero . It seems that charge conjugation is broken in light-cone at the matrix element level. This is a very surprising result since I'd expect charge symmetry to ge a good symmetry of any quantized physical theory. Furthermore, this 'anomalie' can't be fixed by one of the counterterms already discussed, since renormalization theory tells us that a conserved current like the electromagnetic current does not get renormalized - its $Z_J$ is identically 1.

For $\mu = -$ the result is

$$< e^+e^-(p)|_{para} J^-(q)|e^+e^-(l) >_{para} = -2\sqrt{4\pi\alpha} - \sqrt{\pi\alpha}\frac{Q^2}{m^2} =$$

$$-e\int \frac{d^2k_\perp}{16\pi^3}[dx]\{\frac{4}{x_1x_2}(2\frac{k_\perp^2 + m^2 + x_1 k_\perp.q_\perp}{m^2} + \frac{x_2 q_\perp^2}{m^2}\}\phi^*_{para}(k_\perp, x_i)\phi_{para}(k_\perp, x_i)$$

where I have ignored $< \frac{k_\perp^2}{m^2} >$ which is of order $\alpha^2$ and the $k_\perp.q_\perp$ term gives zero as the $\phi$'s are independent of the angle between $k_\perp$ and $q_\perp$ since the m=0 solution are taken.

A similar result follows for $\mu = i$

$$< e^+e^-(p)|_{para} J^i(q)|e^+e^-(l) >_{para} = -\sqrt{\pi\alpha}\frac{q_i}{m} =$$

$$-e\frac{q^i}{m}\int \frac{d^2k_\perp}{16\pi^3}[dx]\{\frac{2}{x_1x_2}\}\phi^*_{para}(k_\perp, x_i)\phi_{para}(k_\perp, x_i)$$

Note that it seems that at this level, rotational invariance is also broken, since the + and - components of $J^\mu$ are not equal:

$$-2\sqrt{4\pi\alpha} \neq -2\sqrt{4\pi\alpha} - \sqrt{\pi\alpha}\frac{Q^2}{m^2}$$

Finally , gauge invariance can e checked by computing the matrix element of the electromagnetic current between states of opposite charge conjugation, so it would not vanish just by charge conjugation. The following is the result

$$q_\mu < e^+e^-(p)|_{ortho} J^\mu(q)|e^+e^-(l) >_{para} = 0 =$$

$$-ie\int \frac{d^2k_\perp}{16\pi^3}[dx]\{\frac{4k_\perp \times q_\perp}{\sqrt{2}x_1x_2m}\}\phi^{0*}_{ortho}(k_\perp, x_i)\phi_{para}(k_\perp, x_i)$$

since the amplitudes $\phi^0_{ortho}$ and $\phi_{para}$ are independent of the angle between $k_\perp$ and $q_\perp$ . So at least gauge invariance is maintained at the matrix element level.

## 5. Conclusions and Comments

Implimenting charge symmetry to study the hyperfine splitting in positronium , two separate equations for the para and ortho state are obtained. At this point it is observed that counterms to the Hamiltonian are needed to fix the singularities due to rotational and gauge symmetries breaking down. The surprising result is that when computing matrix elements of the electromagnetic current, charge symmetry is not respected, nor is rotational invariance. The reader might say that this is not a large effect, and for QED that might be true. But as a question of principle and more specifically for strongly interacting theories like QCD, this is rather problematic. Perhaps we should investigate an alternative approach like Dirac-Bergmann algorithm (DBA)[4] as studied recently by the Regensburg group[5 6 7 8 9 10 11 12]

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