Some thoughts on Algebraic Graph Theory

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Abstract

This paper looks at some aspects of graph theory from an algebraic point of view. The focus of the paper is on automorphisms of the graph onto itself, and at vertex-transitive mappings.

To facilitate the discussion, we introduce some notions from group theory and from category theory as well.

1. Introduction

In this paper we will discuss some aspects of algebraic graph theory, relying on the book by N Biggs [Biggs], but will also make use of concepts from group theory as presented in N Jacobson's book [Jacobson], and also use some ideas from category theory as presented informally by R Geroch [Geroch]. For our graph theory definitions, we will rely on those of the class text, Clark and Holton [Clark & Holton].

In this chapter we will set up our discussion by introducing some ideas from category theory. Category theory can be described informally as the branch of mathematics in which one studies certain definitions in a broader context – mathematics of mathematics, as it were [Geroch, Ch 2].

A category consists of three things:

i) A class O, with elements will be called *objects*
ii) A set Mor(A,B), whose elements will be called *morphisms* from A to B, and A & B are any two objects
iii) A rule which assigns, for any given objects and any morphism φ from A to B and morphism ψ from B to C, a morphism $\varphi \circ \psi$ from A to C. This latter morphism is a *composition* of φ with ψ. It is subject to the following two conditions:

i) Composition is associative. If A, B, C and D are four objects, and φ, ψ, λ are morphisms from A to B, from B to C and from C to D respectively, then

$$(\lambda \circ \psi) \circ \varphi = \lambda \circ (\psi \circ \varphi)$$

ii) Identities exist. For each object A, there is a morphism ι_A from A to A (called the *identity* morphism on A) with the following property: for any morphism φ from A to B, then

$$\varphi \circ \iota_A = \varphi$$

if μ is any morphism from C to A, then

$$\iota_A \circ \mu = \mu$$

This is the definition of a category.

Geroch give examples of many categories, like sets, groups, vector spaces, associative algebras, Lie algebras, topological spaces, and so on.

One category we will be using a lot in this paper is that of groups [Geroch Ch 3]. A *group* consists of two things:

1. A set G.
2. A rule which, given two elements g and g' of G, assigns an element of G (normally written gg'), with the following conditions:

   i) The product is associative. For any three elements g, g', g" of G,

$$g\,(g'g'') = (g\,g')\,g''$$

   ii) An identity exists. There is an element of G, e (the *identity*), with the following property:

$$e\,g = g\,e = g$$

iii) Inverses exist. Given an element g of G, there is an element of G, g^{-1} called the *inverse* of g, such that

$$g\,g^{-1} = g^{-1}\,g = e$$

It turns out that in some of the applications we will be discussing, we will be using matrix representations of graphs. To clarify how this is useful, as well as to give a broader context, let us introduce Geroch's definition of representations [Geroch Ch 22].

The idea of representations is to represent an object as a collection of morphisms. In this manner, objects become more concrete – rather than being abstract elements, they do something, namely act as morphisms. Consider two categories, **C** and **C'.** A *representation* of object A in C consists of an object P' in C'together with a morphism ψ:A–>Mor(P', P') in category C. Thus, for each element a of object A, we must find a certain morphism ψ_a from P' to itself. This morphism is not arbitrary, but must be such that ψ itself is a morphism in category C'.

The object A is represented by a bunch of morphisms from P' to P' with this representation (by which a in A goes to morphism ψ_a from P' to P') reflecting the structure within the object A.

As an example, consider an abstract group G. A representation of G (in the category of the matrix group and matrix multiplication) consists of a group of matrices M, together with a rule which assigns to each element g of G a linear mapping ψ_g from group M to M such that $\psi_{gg'} = \psi_g \circ \psi_{g'}$ and $\psi_e = \iota_{M.}$

2. Linear Algebra and Graph Theory

We will discuss now some results from Biggs' book. First, we will look at some results in linear algebra [Biggs Ch 2].

Let us recall the adjacency matrix A, as in [Clark & Holt, Ch 1], associated with a simple connected graph G. Since it is a matrix, we can define its eigenvalues. Together with their respective multiplicities, we can form the *spectrum* of G: Given the s eigenvalues $\lambda_0 > \lambda_1 .... > \lambda_{s-1}$ and their respective multiplicities $m(\lambda_0)$, $m(\lambda_1)$, ..., $m(\lambda_{s-1})$, the spectrum of G is

$$\text{Spec}(G) = (\lambda_0 , \lambda_1 ,..., \lambda_{s-1} ; m(\lambda_0), m(\lambda_1), ..., m(\lambda_{s-1})).$$

Thus, the complete graph K_4 has the following adjacency matrix A;

$$A = [0\ 1\ 1\ 1;1\ 0\ 1\ 1;\ 1\ 1\ 0\ 1;\ 1\ 1\ 1\ 0]$$

and a spectrum

$$\text{Spec}(K_4) = (\ 3\ –1;\ 1\ 3)$$

We can construct the characteristic polynomial χ associated with the graph G out of its eigenvalues from the eigenvalue equation det $(\lambda I – A) = 0$

$$\chi(G, \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2 \lambda^{n-2} + c_3 \lambda^{n-3} + ... + c_{n.}$$

Proposition 2.3 *The coefficients of the characteristic polynomial of a graph G satisfy the following:*

i) $c_1 = 0$;
ii) $– c_2$ *is the number of edges in G ;*
iii) $– c_3$ *is twice the number of triangles in G.*

Proof For $i \in \{1,2,3,....,n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of A which have *i* rows and columns. Then,
(1) Since A has zero diagonal, $c_1 = 0$.
(2) A principal minor with two rows and columns with non-zero entry must be of the form [0 1; 1 0]. There is one such minor for each pair of adjacent vertices in G, each with value (-1). Hence $(-1)^2 c_2 = - |E(G)|$.

(3) There are essentially three possibilities for non-trivial principal minors with three rows and columns: [0 1 0;1 0 0;0 0 0], [0 1 1;1 0 0;1 0 0] and [0 1 1;1 0 1; 1 1 0]. Of these, only the last is non-zero, with value 2. It corresponds to three mutually adjacent vertices in G, so we get the description of c_3.

Lemma 2.5 *The number of walks on length l in G, from a vertex v_i to v_j, is $(A^l)_{ij}$.*

Proof The result is true for l = 0, since $A^0 = I$ and for l = 1, since $A^1 = A$, the adjacency matrix. Suppose it is true for l = k. The set of walks of length k+1 from v_i to v_j is in bijective correspondence to the set of walks of length k from v_i to v_h adjcent to v_j. Thus, the number of walks is

$$\Sigma_{\{vh,\, vj\}\, \varepsilon\, E(G)} \quad (A^k)_{ih} = \Sigma_{h=1}^{\ n} (A^k)_{ih}\, a_{hj} = (A^{k+1})_{ij}$$

So the number of walks of length k+1 from v_i to v_j is $(A^{k+1})_{ij}$.

Let us defince the algebra of polynomials in the adjacency matrix A the adjacency algebra A(G).

Proposition 2.6 *Let G be a connected graph with adjacency algebra A(G) and diameter d (as defined in class). Then the dimension of A(G) is at least d+1.*

Proof Let x and y be vertices in G such that d(x,y) = *d,* and suppose that

$$x = v_0\, v_1\, ...v_d = y$$

is a walk of length *d*, and v_i ε V(G).

Then for each *i* ε {1,2,3,....,d}, there is at least one walk of length *i,* but no shorter, joining v_0 to v_i.Then A^i has non-zero entry in a position where I, A, A^2, ...A^{i-1} are zero. It follows then that A^i is not linearly dependent on I, A, A^2, ...$A^{i-1,}$ and that { I, A, A^2, ...A^d } are linearly independent in A(G). Since there are *d+1* elements, A(G) has at dimension at least *d+1.*

If the adjacency matrix has *s* distinct eigenvalues, then its minimum polynomial has degree *s*, since A is real symmetric. Therefore the dimension of the A(G) is *s*. We have the following corollary.

Corollary 2.7 *A connected graph G with diameter d has at least d+1 distinct eigenvalues.*

Consider bipartite graphs, as defined in [Clark & Holton, Ch 1]. If we order the vertex set so that those is X come first, followed by those in Y, then the adjacency matrix A can be written as

$$A = [\mathbf{0}\ \ \mathbf{B}\ ;\mathbf{B}^t\ \ \mathbf{0}\ ]$$

where **0** and **B** are matrices.If **x** is an eigenvector with eigenvalue λ, and **y** is obtained from **x** by changing the signs of the entries corresponding to entries in Y, then **y** is an eigenvector with eigenvalue $-\lambda$. It follows that the spectrum of a bipartite graph is symmetric with respect to 0.

Recall that Lemma 2.5 said that the total number of closed walks of length l is equal to tr A^{l}. Since the trace of a matrix is the sum of its eigenvalues, we can express this as

$$\mathrm{tr}\, A^{l} = \Sigma\, \lambda^{l}$$

In particular, the sum of eigenvalues is zero, the sum of squares is twice the number of edges, and the sum of cubes is six times the number of triangles.

3. Regular graphs and line graphs

In this section we will focus on regular graphs. A graph is regular, or *k-regular* if each vertex has degree k.

Proposition 3.1 *Let G be a regular graph of degree k. Then:*

(1) k is an eigenvalue of G;
(2) if G is connected, then the multiplicity of k is 1;
(3) *for any eigenvalue λ of G, we have $|\lambda| <= k$.*

Proof

(1) Let $\mathbf{u} = [1,1,1,....1]^t$**;** then if A is the adjacency matrix of G, we have $\mathbf{Au} = k\mathbf{u}$**,** since there are *k* 1's in each row. Therefore *k* is an eigenvalue of G.

(2) Let $\mathbf{x} = [x_1, x_2,...x_n]^t$ denote any non-zero vector so that $\mathbf{Ax} = k\mathbf{x}$ and suppose that x_j is the entry in **x** with the largest absolute value. Since $(\mathbf{Ax})_j = kx_j$, we have

$$\Sigma' x_i = kx_j$$

where Σ' denotes summation over those k vertices v_i adjacent to v_j. By the maximal property of x_j, we have $x_j = x_i$ for all vertices. If G is connected, we can show then that all entries of **x** are equal. So **x** is a multiple of **u**, hence has multiplicity 1 since the space of eigenvectors of eigenvalue k is 1.

(3) Suppose that $\mathbf{Ay} = \lambda\, \mathbf{y}$, **y** not zero and let y_j denote the entry with the largest absolute value. By the same argument as in (2), we have $\Sigma' y_i = ky_j$, so

$$|\lambda||y_j| = |\Sigma' y_i| <= k\, |y_j|$$

Thus $|\lambda| <= k$, as required.

Let us define now the line graph L(G), similar to the dual graph of Clark & Holton [Clark & Holton, Ch 5]. We take the edges of G as the vertices of L(G), and join two vertices in L(G) whenever the corresponding edges in G have a common vertex. We define an n x m matrix X thus – where $|V(G)| = n$ and $|E(G)| = m$: $(\mathbf{X})_{ij} = 1$ if v_i is incident on e_j, 0 otherwise.

Lemma 3.6 *Suppose G and X are as above. Let A be the adjacency matrix of G, A_L that of L(G). Then:*

(1) $\mathbf{X}^t\mathbf{X} = \mathbf{A}_L + 2\mathbf{I}_m$;
(2) *If G is regular of degree k, then* $\mathbf{X}\,\mathbf{X}^t = \mathbf{A} + k\mathbf{I}_n$;

Proof
(1) We have

$$(\mathbf{X}^t\mathbf{X})_{ij} = \Sigma\, (\mathbf{X})_{li}\, (\mathbf{X})_{lj}$$

from which it follows that $(\mathbf{X}^t\mathbf{X})_{ij}$ is the number of vertices v_l of G which are incident on both edges e_i and e_j. The required result follows from the definition for L(G) and $\mathbf{A}_L$.

(3) This is proven similarly to (1).

Theorem 3.8 If G is a regular graph of degree k with n vertices and m = 1/2 nk edges, then

$$\chi(L(G); \lambda) = (\lambda+2)^{m-n}\, \chi(G; \lambda+2-k)$$

Proof We use notation and result of Lemma 3.6. We define two matrices **U** and **V** as follows:

$$\mathbf{U} = [\lambda\mathbf{I}_n\ \ -\mathbf{X};\ \mathbf{0}\ \mathbf{I}_m] \text{ and } \mathbf{V} = [\mathbf{I}_n\ \mathbf{X};\ \mathbf{X}^t\ \lambda\mathbf{I}_m]$$

Then we have

$$\mathbf{UV} = [\lambda\mathbf{I}_n - \mathbf{X}\,\mathbf{X}^t\ \ \mathbf{0};\ \mathbf{X}^t\ \lambda\mathbf{I}_m\ ] \text{ and } \mathbf{VU} = [\lambda\mathbf{I}_n\ \ 0:\ l\mathbf{X}^t\ \ \lambda\mathbf{I}_m - \mathbf{X}^t\mathbf{X}]$$

Taking determinants of both of these terms, we get det (**UV**) **=** det (**VU**), so we get

$$\lambda^m \det(\lambda\mathbf{I}_n - \mathbf{X}\,\mathbf{X}^t) = \lambda^n \det(\lambda\mathbf{I}_m - \mathbf{X}^t\mathbf{X})$$

Then we have the following:

$$\begin{aligned}
\chi(L(G); \lambda) &= \det(\lambda\mathbf{I}_m - \mathbf{A}_L) \\
&= \det((\lambda+2)\mathbf{I}_m - \mathbf{X}^t\mathbf{X}) \\
&= (\lambda+2)^{m-n} \det((\lambda+2)\mathbf{I}_n - \mathbf{X}\,\mathbf{X}^t) \\
&= (\lambda+2)^{m-n} \det((\lambda+2-k)\mathbf{I}_n - \mathbf{A}) \\
&= (\lambda+2)^{m-n}\, \chi(G; \lambda+2-k)
\end{aligned}$$

Then**,** if the spectrum of G is

$$\mathrm{Spec}(G) = (k, \lambda_1, \ldots, \lambda_{s-1}; 1, m(\lambda_1), \ldots, m(\lambda_{s-1}))$$

The spectrum of L(G) is

$$\mathrm{Spec}(L(G)) = (2k-2, k-2+\lambda_1, \ldots, -2; 1, m(\lambda_1), \ldots, m-n)$$

If G is K_t, then for the line graph $L(K_t)$ we get what is called the triangle graph Δ_t, with vertices corresponding to the 1/2t(t-1) pairs of numbers from {1,2,3,....t}. The $\mathrm{Spec}(\Delta_t)$ is

$$\mathrm{Spec}(\Delta_t) = (2t-4\ \ t-4\ \ -2;\ 1\ \ t-1\ \ 1/2t(t-3))$$

Consider now the complement of G, G^c, similar to Clark & Holton's definition in [Clark & Holton, Ch1, Ex 1.5.2]. We can see then that

$$\mathbf{A} + \mathbf{A}^c = \mathbf{J} - \mathbf{I}$$

where $\mathbf{A}^c$ is the adjacency matrix for G^c and $\mathbf{J}$ is a matrix full of 1's. Using arguments similar to those of Theorem 3.8, it can be shown that

$$(\lambda+k+1)\, \chi(G^c; \lambda) = (-1)^n(\lambda-\nu+\kappa+1)\ \ \chi(G; -\lambda-1)$$

4. Automorphisms of graphs

Automorphism of a simple graph G is a permutation π of V(G), such that if {u,v} ε E(G) iff {π(u), π(v)} also ε E(G). The set of all automorphisms of G, plus the composition operation form the automorphism group Aut(G) [Jacobson, Ch 1]. Two vertices x and y belong to the same *orbit* if there is an automorphism α such that $\alpha(x) = y$. It can be seen then that d(x) = d(y), x and y have the same degree.

G is *vertex-transitive* if Aut(G) acts transitively on V(G), meaning that there is only one orbit. This means that if we have any two vertices u, v, then there is an automorphisms π ε Aut(G) such that $\pi(u) = v$. The action of Aut(G) on V(G) induces an action on E(G) thus: $\pi\{x,y\} = \{\pi(x),\pi(y)\}$. We say that G is *edge-transitive* if this action is transitive, i e, the automorphism maps one edge into another.

Proposition 15.1 *If a connected graph is edge-transitive but not vertex-transitive, then it is bipartite.*

Proof Let {x,y} be and edge of G, and let X and Y be orbits containing x and y respectively under the action of Aut(G). It follows from the definition of an orbit that X and Y are either disjoint or identical. Since G is connected, every vertex z is in some edge {z,w}, and since G is edge-transitive, z belongs to either X or Y. So X U Y = V(G).
If X = Y = V(G), G is vertex-transitive, contradicting the hypothesis. So $X \cap Y = 0$.
Every edge has one end in X and another in Y, so G is bipartite.

Let represent permutations on V(G) by a matrix $P = (p_{ij})$ such that $p_{ij} = 1$ if $v_i=\pi(v_j)$, $p_{ij} = 0$ otherwise.

Proposition 15.2 *Let A be the adjacency matrix of G, and π a permutation of V(G). Then π is an automorphism of G iff PA = AP, where P is the permutation matrix of π.*

Proof Let $v_h = \pi(v_j)$, and $v_k = \pi(v_j)$ Then we have

$$(PA)_{hj} = \Sigma\, p_{hl}\, a_{lj} = a_{ij} \;;$$

$$(AP)_{hj} = \Sigma\, a_{hl}\, p_{lj} = a_{hk}$$

Then, AP = PA iff $\mathbf{v_j}$ and $\mathbf{v_j}$ are adjacent whenever $\mathbf{v_h}$ and $\mathbf{v_k}$ are. That is, iff π is an automorphism of G.

This means that automorphisms produce multiple eigenvectors corresponding to a given eigenvalue: AP x = PA x = Pλ x = λ Px, if x has eigenvalue λ, so does Px.

Lemma 15.3 Let λ a simple eigenvalue of G, with x the corresponding eigenvector, with real components. If the permutation matrix P represents an automorphism of Aut(G), then Px = $\pm$ x.

Proof If λ has multiplicity one, x and Px are linearly dependent; that is, Px = μx, for some μ complex. Since x and P are real, μ is also real. And, since $P^s = I$ for some integer $s \geq 1$, μ is the sth root of unity. Since λ has multiplicity one, $\mu = \pm 1$.

We have now the following theorem:

Theorem 15.4 *If all the eigenvalues of a graph G are simple, every automorphism of G (except for the identity) has order 2.*

Proof Suppose that every eigenvalue of G has multiplicity one. Then, for any permutation matrix P representing an automorphism of G, and for any eigenvector x, we have $\mathbf{P^2\, x = x.}$ The space spanned by the eigenvectors is the whole space of column vectors, so $\mathbf{P^2 = I}$.

For later in our discussion, we introduce two more concepts.

Definition 15.5 Let G be a graph with automorphism group Aut(G). G is *symmetric* if for any vertices u,v,x,y of V(G) such that u adjacent ot v, and x adjacent to y, exists an automorphism α in Aut(G) so that α (u) = x and α (v) = y. We say that G is *distance-transitive* if for all vertices u,v,x,y so that d(u,v) = d(x,y), there is an automorphism α in Aut(G) so that α (u) = x and α (v) = y.

We see that there is the following hierarchy:

distance-transitive => symmetric => vertex-transitive.

Before exploring this in greater depth, let us note a few things about Aut(G). For any value of n, Aut(K_n) contains n! permutations of its vertices, so it is the *symmetric group* S_n This is in agreement with Corollary to Cayley's theorem for groups in [Jacobson, Ch 1]: Any group is isomorphic to a transformation group, and any finite group of order n is isomorphic to a subgroup of the symmetric group $S_{n.}$ So other graph with n vertices different than K_n will be isomorphic to a subgroup of $S_{n.}$

The automorphism groups Aut(G) and Aug(L(G)) need not have a simple relationship. $L(K_2) = K_1$ so that $Aut(K_1)$ is trivial, while $Aut(K_2)$ is not. But there **is** a group homomorphism θ: Aut(G) -> Aug(L(G)) defined by

$$(\theta g)\ \{u,v\} = \{\theta u,\theta v\}, \text{ where } g\ \varepsilon\ Aut(G),\ \{u,v\}\ \varepsilon\ E(G)$$

and

i) θ is a monomorphism if G is not K_2;

ii) θ is a epimorphism provided G is not $K_{4,}$ K_4 with one or two edges deleted.

We end this section with a remark regarding graphs in which Aut(G) act transitively on both edges and vertices. It can be shown then that G is regular, and if its degree is odd, G is symmetric. If the degree is even, the conclusion is a bit murkier.

5. Vertex-transitive graphs

We define first F = Aut(G). The stabilizer subgroup F_v of a vertex v is a subgroup of F containing those automorphisms which fix v. In the vertex-transitive case, all stabilizer subgroups F_v with v ε V(G), are conjugate, and therefore isomorphic. The index of F_v is [Jacobson, Ch 1.12],

$$|F: F_v| = |F| / |F_v| = |V(G)|$$

If each stabilizer F_v is the identity group, then every element of F (except the identity) does not fix any vertex, and we say that F acts *regularly* on V(G). In this case the order of F is equal to the number of vertices n.

There are many vertex-transitive graphs, which can be constructed with a method developed by Cayley (1878). Let G be any abstract finite group with identity 1; suppose Ω is a set of generators of G, such that

i) $x\ \varepsilon\ \Omega \Rightarrow x^{-1}\ \varepsilon\ \Omega$

ii) 1 not in Ω

Definition 16.1 The Cayley graph C = C(H, Ω) is the simple graph whose vertex set and edge set is defined thus:

$$V(C) = H,\ E(C) = \{g,h\}|\ g^{-1}h\ \varepsilon\Omega\}$$

It can be shown that E(C) is well-defined, and that C(H, Ω) is a connected graph. If H is the symmetric group S_3 and Ω = {(12), (23), (13)}, then the Cayley graph C(H, Ω) is isomorphic to $K_{3,3.}$ To see this, consider the partition of V(C) thus: X={1, (132), (123)} and Y = {(13),(23),(12)}. Then mapping will go only from elements from X to Y and from Y to X.

Proposition 16.2 *(1) The Cayley group $C(H, \Omega$ is vertex-transitive.*
(2) Suppose that π is an automorphism of H such that $\pi(\Omega) = \Omega$. Then π, viewed as a permutation on V(C) is a group automorphism fixing the vertex 1.

Proof (1) For each g in H, we may define a permutation g' of V(C) = H by the rule g'(h) = g'h (h ε H). This permutation is an automorphism of C, for

$$\{h,k\} \varepsilon E(C) \Rightarrow h^{-1} k \varepsilon \Omega$$

$$\Rightarrow (gh)^{-1} gk \varepsilon \Omega$$

$$\Rightarrow \{g'(h), g'(k)\} \varepsilon E(C)$$

The set of all g' (g ε H) is a group H', isomorphic to H. H' is also a subgroup of the full group of automorphisms of C(H,Ω), and acts transitively on vertices.

(2) Since π is a group automorphism, it must fix 1. Also, π is graph automorphism, since

$$\{h,k\} \varepsilon E(C) \Rightarrow h^{-1} k \varepsilon \Omega \Rightarrow \pi(h^{-1} k) \varepsilon \Omega$$

$$\Rightarrow \pi(h)^{-1} \pi(k) \varepsilon \Omega$$

$$\Rightarrow \{\pi(h), \pi(k)\} \varepsilon E(C)$$

Lemma 16.3 *Let G be a connected graph. Then a subgroup H of Aut(G) acts regularly on vertices iff G is isomorphic to a Cayley group C(H,Ω), for some set Ω which generates H.*

Proof Suppose $V(G) = \{v_1, v_2, ... v_n\}$, and H is a subgroup of Aut(G) acting regularly on V(C). For $1 <= i <= n$, there is a unique h_i ε H so that $h_i (v_1) = v_i$. Let

$$\Omega = \{ h_i \ \varepsilon H \mid v_i \text{ is adjacent to } v_1 \text{ in } C\}$$

We can check quickly that Ω satisfies requirements for Definition 16.1, and that the bijection $v_i <-> h_i$ isa graph automorphism of G with C(H,Ω).

Conversely, if G = C(H,Ω), then the group H' defined in the proof of Proposition 16.2 acts regularly on V(G) and H' is isomorphic to H.

Lemma 16.3 shows then that if Aut(G) itself acts regularly on V(G), then G is a Cayley graph C(Aut(G), Ω).

Proposition 16.5 *Let G be a vertex-transitive graph whose automorphism group F=Aut(G) is abelian. Then F acts regularly on V(G), and F is an elementary abelian 2-group.*

Proof If g and h are elements of the abelian group F, and g fixes v, then gh(v) = hg(v) =h(v) so g fixes h(v) as well. If F is transitive, every vertex is of the form h(v) for some h in F, so g fixes every vertex. Hence g=1.

Thus F acts regularly on V(G), so by Lemma 16.3, G is a Cayley graph C(F, Ω). Now since F is Abelain, the map $g \to g^{-1}$ is an automorphism of F, at it fixes Ω setwise. Suppose this automorphism is not trivial. Then by part (2) of Proposition 16.2, this implies that F is not regular. But this is a contradiction. So, $g = g^{-1}$ for all g in F, so F is of order 2.

Proposition 16.6 *Let G be a vertex-transitive graph with degree k, and let λ be a simple eigenvalue of G. If $|V(G)|$ is odd, then $\lambda = k$, If $|V(G)|$ is even, λ is one of the integers $2\alpha - k$, where $0 <= \alpha <= k$.*

Proof Let **x** be a real eigenvector corresponding to the simple eigenvalue λ, and let **P** be a permuation matrix representing an automorphism π of G. If $\pi(v_i) = v_j$ then by Lemma 15.3,

$$x_i = (\mathbf{Px})_j = \pm x_j$$

Since G is vertex-transitive, we obtain that all the entries of **x** have the same absolute value. We know from previous results that $\mathbf{u} = [1,1,1,...,1]^t$ is an eigenvector with eigenvalue k. If λ is not k, we must have $\mathbf{u}^t\mathbf{x} = 0$. Which means $\Sigma x_i = 0$. This is not possible if there are odd summands.

If G has an even number of vertices, choose a vertex v_i of G and suppose α adjacent vertices v_j have $x_j = x_i$, while $k - \alpha$ have $x_j = - x_i$. Since $(\mathbf{Ax})_i = \lambda x_i$, we get $\Sigma' x_j = \lambda x_i$ and the sum is over vertices adjacent to v_j. We obtain

$$\alpha x_i - (k - \alpha) x_i = \lambda x_i$$

so $\lambda = 2\alpha - k$.

Let us strengthen the assumptions by demanding that G be symmetric. The simple eigenvalues can be restricted even more.

Proposition 16.7 *Let G be a symmetric graph of degree k, and let λ be a simple eigenvalue of G. Then $\lambda = \pm k$.*

Proof We continue with the notation of the previous Proposition. Let v_j and v_l be any two vertices adjacent to v_i; then there exists an automorphism π of G so $\pi(v_i) = v_j$ and $\pi(v_j) = v_l$. If **P** is the permutation matrix representing π, $\pi(v_i) = v_j$ implies $\mathbf{Px} = \mathbf{x}$ and $x_j = x_l$. Thus $\alpha = 0$ or k, so $\lambda = \pm k$.

Note that the eigenvalue –k occurs and is simple iff G is bipartite.

Suppose now that G is a vertex-transitive graph with F= Aut(G). For any vertex v of G, define

$$L_v = \{g \in F \mid g \text{ fixes each vertex adjacent ot } v\}$$

Then L_v is a normal subgroup of F_v [Jacobson Ch 1.8] - a subgroup L of F is normal or self-conjugate if $f^{-1} l f \in F$ for every $f \in F$ and $l \in L$. By the fundemantal theorem of group homomorphisms, there is a homomorphism from F_v into the group of all permutations of the neighbors of v, with kernel L_v - kernel is the inverse of the automorphic map of the identity. Therefore, $| F_v : L_v | <= k!$, where k is the degree.

Let G be connected vertex-transitive graph and let F_v denote the stabilizer of vertex v. If h is any automorphism of G for which $d(v,h(v)) = 1$, and G is symmetric, then h and F_v generate Aut(G).

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